Nonlinear Analysis: Real World Applications 45 (2019) 26–52

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

Global weak solutions for the three-dimensional chemotaxis-Navier–Stokes system with slow p-Laplacian diffusion

Weirun Tao, Yuxiang Li*

Institute for Applied Mathematics, School of Mathematics, Southeast University, Nanjing 211189, PR China

ARTICLE INFO

Article history:
Received 4 December 2017
Received in revised form 8 June 2018
Accepted 9 June 2018

Keywords:
Chemotaxis
Navier–Stokes equation
Nonlinear diffusion
Global existence

ABSTRACT

This paper investigates an incompressible chemotaxis-Navier–Stokes system with slow p-Laplacian diffusion

$$\begin{cases} n_t + u\cdot\nabla n = \nabla\cdot(|\nabla n|^{p-2}\nabla n) - \nabla\cdot(n\chi(c)\nabla c), & x\in\Omega,\ t>0,\\ c_t + u\cdot\nabla c = \Delta c - nf(c), & x\in\Omega,\ t>0,\\ u_t + (u\cdot\nabla)u = \Delta u + \nabla P + n\nabla\Phi, & x\in\Omega,\ t>0,\\ \nabla\cdot u = 0, & x\in\Omega,\ t>0 \end{cases}$$

under homogeneous boundary conditions of Neumann type for n and c, and of Dirichlet type for u in a bounded convex domain $\Omega\subset\mathbb{R}^3$ with smooth boundary. Here, $\Phi\in W^{1,\infty}(\Omega)$, $0<\chi\in C^2([0,\infty))$ and $0\le f\in C^1([0,\infty))$ with $f(0)=0$. It is proved that if $p>\frac{32}{15}$ and under appropriate structural assumptions on f and χ, for all sufficiently smooth initial data (n_0,c_0,u_0) the model possesses at least one global weak solution.

1. Introduction

In this paper, we consider the following chemotaxis-Navier–Stokes system with p-Laplacian diffusion

$$\begin{cases} n_t + u\cdot\nabla n = \nabla\cdot\left(|\nabla n|^{p-2}\nabla n\right) - \nabla\cdot(n\chi(c)\nabla c), & x\in\Omega,\ t>0,\\ c_t + u\cdot\nabla c = \Delta c - nf(c), & x\in\Omega,\ t>0,\\ u_t + \kappa(u\cdot\nabla)u = \Delta u + \nabla P + n\nabla\Phi, & x\in\Omega,\ t>0,\\ \nabla\cdot u = 0, & x\in\Omega,\ t>0 \end{cases} \tag{1.1}$$

in a smooth bounded domain $\Omega\subset\mathbb{R}^3$, where the scalar functions $n=n(x,t)$ and $c=c(x,t)$ denote bacterial density and the concentration of oxygen, respectively. The vector $u=(u_1,u_2,u_3)$ is the fluid velocity field and the associated pressure is denoted by $P=P(x,t)$. The function χ represents the chemotactic sensitivity, f is

* Corresponding author.
E-mail addresses: taoweiruncn@163.com (W. Tao), lieyx@seu.edu.cn (Y. Li).

https://doi.org/10.1016/j.nonrwa.2018.06.005

the oxygen consumption rate by the bacteria and $\kappa \in \mathbb{R}$ measures the strength of nonlinear fluid convection. The given function Φ stands for the gravitational potential produced by the action of physical forces on the cell. If $p > 2$, the nonlinear diffusion $|\nabla n|^{p-2}\nabla n$ is called the slow p-Laplacian diffusion, whereas $1 < p < 2$, it is called the fast p-Laplacian diffusion.

The Keller–Segel model was first presented in [1] to describe the chemotaxis of cellular slime molds. Let n denote the cell density and c describe the concentration of the chemical signal secreted by cells. The mathematical model reads as

$$\begin{cases} n_t = \Delta n - \nabla \cdot (n \nabla c), & x \in \Omega,\ t > 0, \\ c_t = \Delta c - c + n, & x \in \Omega,\ t > 0, \end{cases} \tag{1.2}$$

which is called Keller–Segel system. It is known that whether the classical solutions of the system exist globally depends on the size of initial data (cf. [2–5]). A large number of variants of the classical form have been investigated, including the system with the logistic terms (see [6–8], for instance), two-species chemotaxis system (see [9–12], for instance), attraction–repulsion chemotaxis system (see [13,14], for instance) and so on. We refer to [15–18] for the further reading.

The chemotaxis-Navier–Stokes system was first introduced in [19]. Aerobic bacteria such as *Bacillus subtilis* often live in thin fluid layers near solid–air–water contact line, in which the biology of chemotaxis, metabolism, and cell–cell signaling is intimately connected to the physics of buoyancy, diffusion, and mixing [19]. Both bacteria and oxygen diffuse through the fluid, and they are also transported by the fluid (cf. [20] and [21]). Taking into account all these processes, in [19] the authors proposed the model

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (n \chi(c) \nabla c), & x \in \Omega,\ t > 0, \\ c_t + u \cdot \nabla c = \Delta c - n f(c), & x \in \Omega,\ t > 0, \\ u_t + \kappa (u \cdot \nabla) u = \Delta u + \nabla P + n \nabla \Phi, & x \in \Omega,\ t > 0, \\ \nabla \cdot u = 0, & x \in \Omega,\ t > 0 \end{cases} \tag{1.3}$$

in a domain $\Omega \subset \mathbb{R}^d$, where the vector $u = (u_1(x,t), u_2(x,t), \ldots, u_d(x,t))$ is the fluid velocity field and the associated pressure is represented by $P = P(x,t)$.

The chemotaxis fluid system has been studied in the last few years. In [21], local-in-time weak solutions were constructed for a boundary-value problem of (1.3) in the three-dimensional setting. In [22], global classical solutions near constant states were constructed with $\Omega = \mathbb{R}^3$. For the chemotaxis-Navier–Stokes system in two space dimensions, the authors in [23] obtained global existence for large data. For the case of bounded domain $\Omega \subset \mathbb{R}^d$, Winkler [24] proved that the initial–boundary value problem of (1.3) possesses a unique global classical solution for $d = 2$ and possesses at least one global weak solution for $d = 3$ under the assumption that $\kappa = 0$. In [25] the same author showed that in bounded convex domains $\Omega \subset \mathbb{R}^2$, the global classical solutions obtained in [24] stabilize to the spatially uniform equilibrium $(\bar{n}_0, 0, 0)$ with $\bar{n}_0 := \frac{1}{|\Omega|}\int_\Omega n_0(x)dx$ as $t \to \infty$. Recently, Zhang and Li [26] proved that such solution converges to the equilibrium $(\bar{n}_0, 0, 0)$ exponentially in time. By deriving a new type of entropy-energy estimate, Jiang et al. [27] generalized the result of [25] to general bounded domains. (If both χ and f are supposed to be nonnegative and nondecreasing, it was shown by Chae, Kang and Lee [28] that the Cauchy problem of (1.3) admits a global classical solution under the assumption that $d = 2$ and $\sup_c |\chi(c) - \mu f(c)|$ be sufficiently small for some $\mu > 0$. It was showed in [29] that the 2-dimensional Cauchy problem of (1.3) admits global classical bounded solutions for regular initial data. For more results of the well-posedness of the Cauchy problem to (1.3) in the whole space we refer the reader to [22,23,30–32].)

The diffusion of bacteria sometimes depend nonlinearly on their densities [16,33–35]. Introducing this into the model (1.3) leads to the chemotaxis-Navier–Stokes system with nonlinear diffusion [36]

$$\begin{cases} n_t + u \cdot \nabla n = \nabla \cdot (D(n) \nabla n) - \nabla \cdot (n \chi(c) \nabla c), & x \in \Omega,\ t > 0, \\ c_t + u \cdot \nabla c = \Delta c - n f(c), & x \in \Omega,\ t > 0, \\ u_t + \kappa (u \cdot \nabla) u = \Delta u + \nabla P + n \nabla \Phi, & x \in \Omega,\ t > 0, \\ \nabla \cdot u = 0, & x \in \Omega,\ t > 0. \end{cases} \tag{1.4}$$