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The Cauchy problem for the dissipative Boussinesq equation[☆]

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ABSTRACT

This work is devoted to the solvability and finite time blow-up of solutions of the Cauchy problem for the dissipative Boussinesq equation in all space dimension. We prove the existence and uniqueness of local mild solutions in the phase space by means of the contraction mapping principle. By establishing the time-space estimates of the corresponding Green operators, we obtain the continuation principle. Under some restriction on the initial data, we further study the results on existence and uniqueness of global solutions and finite time blow-up of solutions with the initial energy at three different level. Moreover, the sufficient and necessary conditions of finite time blow-up of solutions are given.

1. Introduction

In this paper we are concerned with the following Cauchy problem for the dissipative Boussinesq equation

$$u_{tt} - \Delta u + \Delta^2 u - \alpha \Delta u_t + \gamma \Delta^2 u_t + \Delta f(u) = 0, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+ \tag{1.1}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x). \tag{1.2}$$

where $f(u) = \beta |u|^{p-1} u$, $\beta \in \mathbb{R} \setminus \{0\}$, $1 < p < \infty$, the constants $\alpha \geq 0$ and $\gamma > 0$, the function $u = u(x,t)$ denotes the unknown function, Δ is the n-dimensional Laplace operator, the subscript t indicates the partial derivative with respect to t, u_0 and u_1 are the given initial value functions.

Boussinesq [1] derived the first generalized wave equation for the flow in shallow inviscid layer

$$u_{tt} - u_{xx} + \gamma u_{xxxx} = \beta (u^2)_{xx}, \tag{1.3}$$

where the constant coefficients γ and β depend on the depth of fluid and the characteristic speed of long waves. This was the first to give a scientific explanation of the existence to solitary waves found by Scott

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Russell [2], which are long, shallow, water waves of permanent form. Since then, extensive research has been carried out to explore the properties and solutions of the equation and its associated initial (boundary) value problems. Clarkson [3] proposed a general approach for constructing exact solutions of (1.3). Hirota [4] deduced conservation laws and examined the numerical solution of the Boussinesq equation. Yajima [5] investigated the nonlinear evolution of linearly stable solution for Eq. (1.3).

A generalization of one of the Boussinesq type equations is considered which arises in the modeling of nonlinear strings, namely,

$$u_{tt} - u_{xx} + (u_{xx} + f(u))_{xx} = 0. \tag{1.4}$$

In the recent years, Boussinesq equation and its generations have been extensively studied by many mathematicians. Bona and Sachs [6] studied the Cauchy problem of Eq. (1.4). By using Kato's abstract theory of quasilinear evolution equation, they proved the existence of local $H^{s+2} \times H^s$ solution for $f \in C^\infty$ with $f(0) = 0$ and for any $(u_0, u_1) \in H^{s+2} \times H^s$ with $s > \frac{1}{2}$. For $f(u) = |u|^{p-1}u$, $1 < p < 5$, they proved the global existence of $H^{s+2} \times H^s$ solution under some assumptions on initial data. Linares [7] established the local well posedness of the Cauchy problem of Eq. (1.4) with $f(u) = |u|^\alpha u$ for $H^1 \times L^2$ solution when $\alpha > 0$ and for $L^2 \times H^{-1}$ solution when $0 < \alpha < 4$, respectively. He also proved that these local solutions can be extended globally when the size of the data is small. Farah [8] studied the local well posedness of the Cauchy problem of Eq. (1.4) with $f(u) = u^2$ in the space H^s for negative indices of s with $s > -\frac{1}{4}$. The local/global existence, and finite time blow-up of solutions to the Cauchy problem of (1.4) was also established by Tsutsumi and Matahashi [9], Xue [10] and Liu [11], also see [12–15]. For global existence and scattering of solutions, Liu [16] established the global existence and scattering of small amplitude solutions for the initial value problem of (1.4) with $f(u) = |u|^{p-1}u$, and by using potential well method the author obtained invariant sets of solutions and proved the global existence and finite time blow-up of solutions. Cho and Ozawa [17] studied the multidimensional version of (1.4) and their results improved the ones obtained in [7]. Later, some results of [17] were improved by Ferreira [18], in which, the author studied the initial value problem for the n-dimensional ($n \geq 1$) generalized Boussinesq equation (1.4) and proved existence of local and global solutions with singular initial data in weak-L^p spaces. Moreover, long time behavior results and a scattering theory were also obtained.

Boussinesq equation presents in an appropriate balance between the nonlinearity and the dispersion because of the existence of solitary-wave solutions (see [19]). Dissipation is naturally introduced in fluid dynamics through viscosity processes and presence of dissipation in Boussinesq equation generally destroys the balance between nonlinearity and dispersion. However, if an energy production is allowed in the model, another version of Boussinesq Paradigm can be encountered: balance between the energy input and dissipation modulated by the presence of nonlinearity and dispersion,

$$u_{tt} = \left[\gamma^2 u - \frac{\alpha}{2}u^2 - \beta\gamma u_{xx} - \alpha_4 u_{xxt} - \alpha_2 u_t\right]_{xx} \tag{1.5}$$

which encompasses the oscillations of elastic beams (see [20–22]). Here α is the amplitude coefficient, γ is the phase speed of the small disturbances, $(\beta\gamma)$ is the dispersion coefficient, α_2 is the coefficient of energy-production term, and α_4 is the dissipation coefficient.

When $\alpha_4 = 0$, Eq. (1.5) can be rewritten as the damped Boussinesq equation

$$u_{tt} - u_{xx} - 2bu_{xxt} + \alpha u_{xxxx} + \beta(u^2)_{xx} = 0 \tag{1.6}$$

by suitable rescaling the independent variables x and t. Varlamov [23] obtained a classical solution to the Cauchy problem of (1.6) with small initial data by means of the application of both the spectral and perturbation theories.