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Well-posedness of an evolution problem with nonlocal diffusion[☆]

Gonzalo Galiano

Department of Mathematics, Universidad de Oviedo, c/ Federico García Lorca 18, 33007 - Oviedo, Spain

A R T I C L E I N F O

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We prove the well-posedness of a general evolution reaction–nonlocal diffusion problem under two sets of assumptions. In the first set, the main hypothesis is the Lipschitz continuity of the range kernel and the bounded variation of the spatial kernel and the initial datum. In the second set of assumptions, we relax the Lipschitz continuity of the range kernel to Hölder continuity, and assume monotonic behavior. In this case, the spatial kernel and the initial data can be just integrable functions. The main applications of this model are related to the fields of Image Processing and Population Dynamics.

1. Introduction

In this article, we study the well-posedness of a general class of evolution reaction–nonlocal diffusion problems expressed in the following form. Let $T > 0$ and $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ be an open and bounded set with Lipschitz continuous boundary. Find $u : [0, T] \times \overline{\Omega} \to \mathbb{R}$ such that

$$\partial_t u(t, \mathbf{x}) = \int_\Omega J(\mathbf{x} - \mathbf{y}) A(t, \mathbf{x}, \mathbf{y}, u(t, \mathbf{y}) - u(t, \mathbf{x})) d\mathbf{y} + f(t, \mathbf{x}, u(t, \mathbf{x})), \tag{1}$$

$$u(0, \mathbf{x}) = u_0(\mathbf{x}), \tag{2}$$

for $(t, \mathbf{x}) \in Q_T = (0, T) \times \Omega$, and for some $u_0 : \Omega \to \mathbb{R}$.

The main examples we have on mind are connected to the fields of Population Dynamics and of Image Processing. In the first case, choosing for instance $A(t, \mathbf{x}, \mathbf{y}, s) = s$, we describe the balance of population coming in and leaving from $\mathbf{x}$, as

$$\int_\Omega J(\mathbf{x} - \mathbf{y}) u(t, \mathbf{y}) d\mathbf{y} - u(t, \mathbf{x}),$$

where the convolution kernel $J \geq 0$, with $\int J = 1$, determines the size and the shape of the influencing neighborhood of $\mathbf{x}$. In absence of a reaction term, the resulting equation is a nonlocal diffusion variant of

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E-mail address: galiano@uniovi.es.

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the heat equation, usually written as

$$\partial_t u(t,\mathbf{x}) = \int_\Omega J(\mathbf{x}-\mathbf{y})\big(u(t,\mathbf{y})-u(t,\mathbf{x})\big)d\mathbf{y}.$$

In this context, the nonlocal p-Laplacian diffusion operator, corresponding to $A(t,\mathbf{x},\mathbf{y},s) = |s|^{p-2}s$, for $p\in[1,\infty]$, is also a well known example, leading to the equation

$$\partial_t u(t,\mathbf{x}) = \int_\Omega J(\mathbf{x}-\mathbf{y})|u(t,\mathbf{y})-u(t,\mathbf{x})|^{p-2}\big(u(t,\mathbf{y})-u(t,\mathbf{x})\big)d\mathbf{y}.$$

These two examples correspond to a choice for which A is a non-decreasing function of s. These kinds of problems have been studied at great length by Andreu et al. in a series of works, see the monograph [1]. Their results, strongly dependent on the monotonicity of A, include the well-posedness as well as properties such as the stability with respect to the initial data or the convergence of related rescaled nonlocal problems to their corresponding local versions. It is worth mentioning that problems of the type (1) related to monotone functions, A, can be seen as gradient descents of convex energies. For instance, for the p-Laplacian, the nonlocal energy is given by

$$J_p(u) = \frac{1}{p}\int_\Omega\int_\Omega J(\mathbf{x}-\mathbf{y})|u(t,\mathbf{y})-u(t,\mathbf{x})|^p.$$

In the examples arising in Image Processing, the monotonicity of A is not the rule. A very useful denoising filter, the bilateral filter [2–5], which provides results similar to the Perona–Malik equation [6,7] or to the Total Variation restoration filter [8,9], takes the form

$$Bu(\mathbf{x}) = \frac{1}{C(\mathbf{x})}\int_\Omega \exp\Big(-\frac{|\mathbf{x}-\mathbf{y}|^2}{\rho^2}\Big)\exp\Big(-\frac{|u(\mathbf{x})-u(\mathbf{y})|^2}{h^2}\Big)u(\mathbf{y})d\mathbf{y},$$

where u is the image to be filtered, ρ and h are constants modulating the sizes of the space and range neighborhoods where the filtering process takes place, and C is the normalizing factor

$$C(\mathbf{x}) = \int_\Omega \exp\Big(-\frac{|\mathbf{x}-\mathbf{y}|^2}{\rho^2}\Big)\exp\Big(-\frac{|u(\mathbf{x})-u(\mathbf{y})|^2}{h^2}\Big)d\mathbf{y}.$$

Neighborhood filters like B may also be derived from variational principles [10], being their correspondent gradient descent approximations given by nonlocal equations of the type (1). Indeed, defining

$$J(\mathbf{x}) = \exp\Big(-\frac{|\mathbf{x}|^2}{\rho^2}\Big),\quad A(t,\mathbf{x},\mathbf{y},s) = \exp\Big(-\frac{s^2}{h^2}\Big), \tag{3}$$

we have that (1) is the gradient descent associated to the nonconvex energy functional

$$J_B(u) = \int_\Omega\int_\Omega \exp\Big(-\frac{|\mathbf{x}-\mathbf{y}|^2}{\rho^2}\Big)\Big(1-\exp\Big(-\frac{|u(\mathbf{x})-u(\mathbf{y})|^2}{h^2}\Big)\Big)d\mathbf{x}d\mathbf{y},$$

for which the filter $Bu(\mathbf{x})$ is just a one step algorithm in the search direction.

From the definition of A given in (3), we readily see its lack of monotonicity. Thus, the approach followed by Andreu et al. may not be employed to show the well-posedness of the related gradient descent problem.

Besides, there are other situations that we would like to cover for this kind of nonlocal diffusion problems which have been not treated, as far as we know, in the literature. One of them is allowing the convolution kernel, J, to be discontinuous. This is the case we encounter for the Yaroslavsky filter [11], with much faster numerical implementations than that of (3), see [12–14], which is given by

$$J(\mathbf{x}) = 1_{B_\rho(\mathbf{x})}(\mathbf{y}),\quad A(t,\mathbf{x},\mathbf{y},s) = \exp\Big(-\frac{s^2}{h^2}\Big), \tag{4}$$

where $1_{B_\rho(\mathbf{x})}$ is the characteristic function of the ball $B_\rho(\mathbf{x})$.

Another situation we are interested in is that in which the power, p, of the p-Laplacian is not constant, which finds applications in image restoration. Two important examples are the following: