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The critical fractional Schrödinger equation with a small superlinear term ☆

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ABSTRACT

In this paper, we study the critical fractional Schrödinger equation with a small superlinear term. By using the Nehari manifold and the Lusternik–Schnirelmann category, we obtain two multiplicity results. Also, without the Ambrosetti–Rabinowitz condition or the monotonicity condition, we prove an existence result.

1. Introduction

The fractional Schrödinger equation is formulated by Laskin [1,2]. It is a fundamental of fractional quantum mechanics. Also, it appears in various areas such as plasma physics, optimization, finance, free boundary obstacle problems, population dynamics and minimal surfaces. For more background, the authors may see [3] and the references therein.

There have been a lot of papers focusing on the fractional Schrödinger equation:

$$(-\Delta)^s u + V(x)u = f(x,u) \quad \text{in} \quad \mathbb{R}^N, \tag{1.1}$$

where $N \geq 3$, $s \in (0,1)$, $(-\Delta)^s$ is the fractional Laplacian operator. In the remarkable paper [4], Caffarelli and Silvestre expressed the nonlocal operator $(-\Delta)^s$ as a Dirichlet-to-Neumann map for a certain elliptic boundary value problem with local differential operators defined on the upper half space. Subsequently, the technique of [4] has been widely used to deal with the fractional equation in the respects of regularity and variational methods. When $V \equiv 1$ and f has subcritical growth satisfying the Ambrosetti–Rabinowitz condition, Felmer, Quaas and Tan [5] proved the existence, regularity, decay and symmetry properties of

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positive solutions of (1.1). When $\lim_{|x|\to\infty} V(x) = +\infty$ and $f(x,u) = |u|^{p-1}u$, where $p \in (1, 1+\frac{4s}{N})$, Cheng [6] obtained the existence of ground state solutions of (1.1). By using the Nehari manifold method, Secchi [7] provided a generalization of the main result in [6]. When f is asymptotically linear at infinity, Chang [8] proved the existence of ground state solutions. Recently, there has been a lot of interest in the study of the singularly perturbed problem of the fractional equation. See, for instance, [9–16] and the references therein.

In this paper, we study the following critical fractional Schrödinger equation:

$$(-\Delta)^s u + V(x)u = \lambda f(x,u) + g(x)|u|^{2_s^*-2}u \ \text{ in } \ \mathbb{R}^N, \tag{1.2}$$

where $N \geq 3$, $s \in (0,1)$, $2_s^* = \frac{2N}{N-2s}$ is the critical exponent. We first study multiplicity of nontrivial solutions of (1.2) when $\lambda > 0$ is small. To our best knowledge, even in the subcritical case, there are few results on this problem. Recently, Bisci and Rădulescu [17] considered the subcritical problem:

$$(-\Delta)^s u + V(x)u = \lambda f(x,u) \ \text{ in } \ \mathbb{R}^N, \tag{1.3}$$

where f has a superlinear behavior at the origin and a sublinear decay at infinity. By using variational methods, they proved that (1.3) has at least two nontrivial solutions in a suitable range of λ. In [18], Quaas and Xia used the Nehari manifold and the Lusternik–Schnirelmann category to obtain multiple solutions for the critical fractional problem in a bounded domain. However, some key steps in [18] rely on the boundness of the domain. In this paper, by using the Lusternik–Schnirelmann category and developing some variational techniques, we obtain two interesting multiplicity results.

Now we state the results. We assume that V, f, g satisfy the following conditions:

- (V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $\inf_{x\in\mathbb{R}^N} V(x) := V_0 > 0$.
- (f_1) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and there exist $p_0 \in (2, 2_s^*)$, nonnegative continuous functions a, b on $\mathbb{R}^N$ such that $|f(x,u)| \leq a(x)|u| + b(x)|u|^{p_0-1}$ for $(x,u) \in \mathbb{R}^N \times \mathbb{R}$. Moreover, $\frac{a(x)}{V(x)}$ is bounded and $\lim_{R\to+\infty} \sup_{|x|\geq R} \frac{b(x)^{2_s^*-2}}{V(x)^{2_s^*-p_0}} = 0$.
- (f_2) For any $x \in \mathbb{R}^N$, $\frac{f(x,u)}{u}$ is positive for $u \neq 0$, nonincreasing on $(-\infty, 0)$, nondecreasing on $(0, +\infty)$.
- (f_3) $\lim_{u\to+\infty} \frac{F(x,u)}{|u|^2} = +\infty$ locally uniformly in $x \in \mathbb{R}^N$ when $N \geq 4$, or $N = 3$ with $s \in (0, \frac{3}{4})$; $\lim_{u\to+\infty} \frac{F(x,u)}{|u|^2 \ln u} = +\infty$ locally uniformly in $x \in \mathbb{R}^N$ when $N = 3$ with $s = \frac{3}{4}$; $\lim_{u\to+\infty} \frac{F(x,u)}{|u|^{2_s^*-2}} = +\infty$ locally uniformly in $x \in \mathbb{R}^N$ when $N = 3$ with $s \in (\frac{3}{4}, 1)$, where $F(x,u) = \int_0^u f(x,s)\mathrm{d}s$.
- (g_1) $g \in C(\mathbb{R}^N, \mathbb{R})$ and there exist $0 < g_0 < g_M$ such that $g_0 \leq g(x) \leq g_M$ for $x \in \mathbb{R}^N$.
- (g_2) There exists $\rho_0 > 0$ such that $g(x) = g_M$ for $\rho_0 < |x| < 2\rho_0$. Moreover, $g(0) < g_M$.

Theorem 1.1. *Assume (V_1), (f_1)–(f_3) and (g_1)–(g_2). Then there exists $\lambda_0 > 0$ such that problem (1.2) has at least two nontrivial solutions for $\lambda \in (0, \lambda_0)$.*

When $\liminf_{|x|\to\infty} g(x) < g_M$, we get the counterpart of Theorem 1.1. Instead of (g_2), we assume the following condition:

- (g_3) The set $\wedge := \{y \in \mathbb{R}^N : g(y) = g_M\}$ is nonempty and bounded. Moreover, there exists $\rho > 0$ such that $g(x) - g(y) = O(|x-y|^\rho)$ uniformly for $y \in \wedge$ in the limit $x \to y$, where $\rho = 2s$ if $N \geq 4$, or $N = 3$ with $s \in (0, \frac{3}{4}]$; $\rho = N - 2s$ if $N = 3$ with $s \in (\frac{3}{4}, 1)$. Moreover, $\liminf_{|x|\to\infty} g(x) < g_M$.

For $d > 0$, let $\wedge_d = \{x \in \mathbb{R}^N : \mathrm{dist}(x, \wedge) < d\}$.

Theorem 1.2. *Assume (V_1), (f_1)–(f_3), (g_1) and (g_3). Then for any $d > 0$, there exists $\lambda_d > 0$ such that problem (1.2) has at least $\mathrm{cat}_{\wedge_d}(\wedge)$ nontrivial solutions for $\lambda \in (0, \lambda_d)$.*