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Global existence of solutions for an m-component cross-diffusion system with a 3-component case study

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ABSTRACT

In this paper, we examine a general m-component reaction–diffusion matrix with a full diffusion matrix and polynomially growing reaction terms through its diagonalization. We establish the invariant regions of the system and derive the necessary conditions for the existence of solutions. The 3×3 case is taken as a case study, where we determine the exact conditions for the positivity of the eigenvalues, which is necessary for the diagonalization process. Numerical examples are used to illustrate and confirm the findings of this paper.

1. Introduction

The present paper is concerned with the global existence of solutions for a reaction–diffusion system with a full diffusion matrix and polynomial growth. The main motivation behind this work is the fact that most reaction–diffusion systems found in the literature assume that the diffusion matrix is a diagonal one, meaning that the spatial dispersion of every species in a certain region is only a result of the same species' concentration gradient (self-diffusion). Although assumed by many due to the fact that it greatly simplifies the calculations and proofs, it may not be realistic in many scenarios. Some recent studies including [1] have shown that in many cases, the diffusion of one species due to a concentration gradient in another (cross-diffusion) is considerable and may even surpass the self-diffusion.

In this paper, we consider the general system given by

$$\frac{\partial U}{\partial t} - A\Delta U = F(U) \text{ in } \Omega \times (0, +\infty), \tag{1.1}$$

with boundary conditions:

$$\alpha U + (1-\alpha)\partial_\eta U = B \text{ on } \partial\Omega \times (0, +\infty), \tag{1.2}$$

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or

$$\alpha U + (1-\alpha) A \partial_\eta U = B \text{ on } \partial\Omega \times (0, +\infty), \tag{1.3}$$

and initial data:

$$U(0, x) = U_0(x) \text{ on } \Omega. \tag{1.4}$$

In the context of this paper, Ω is an open bounded domain of class C^1 in $\mathbb{R}^N$ with boundary $\partial\Omega$, $\frac{\partial}{\partial\eta}$ denotes the outward normal derivative on $\partial\Omega$. We define the vectors U, F, and B trivially as

$$U := (u_1, \ldots, u_m)^T,$$
$$F := (f_1, \ldots, f_m)^T,$$
$$B := (\beta_1, \ldots, \beta_m)^T.$$

The matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{pmatrix} \tag{1.5}$$

contains the real diffusion coefficients of the system. The matrix A^T is assumed to be diagonalizable with positive distinct eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k$ and eigenvectors V_1, V_2, $\ldots$,V_m with

$$V_\ell = (v_{1\ell}, \ldots, v_{m\ell})^T$$

for $\ell = 1, \ldots, m$. Note that the eigenvalues of A^T are identical to those of A. However, the eigenvectors are different. It follows that the determinant equals

$$\det\left(A^T\right) = \prod_{\ell=1}^{k} \lambda_\ell^{m_\ell},$$

where m_ℓ denotes the algebraic multiplicity corresponding to eigenvalue λ_ℓ. Obviously, the sum of the multiplicities must be equal to the number of columns in A^T, i.e.

$$\sum_{\ell=1}^{k} m_\ell = m.$$

For notational purposes, let us define the eigenvectors associated with the ℓth distinct eigenvalue λ_ℓ as

$$V_{\sigma_\ell+1}, V_{\sigma_\ell+2}, \ldots, V_{\sigma_\ell+m_\ell}, \ \ell = 1, \ldots, k,$$

with

$$\sigma_\ell = \begin{cases} 0 & \ell = 1 \\ \sum_{i=1}^{\ell-1} m_i & \ell = 2, \ldots, k. \end{cases}$$

The eigenvectors are arranged into the matrix P defined as

$$P = \left((-1)^{i_1} V_1 \ \ (-1)^{i_2} V_2 \ \ \ldots \ \ (-1)^{i_m} V_m\right), \tag{1.6}$$