Nonlinear Analysis: Real World Applications 45 (2019) 285–298

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

Energy dependent potential problems for the one dimensional p-Laplacian operator

Hikmet Koyunbakan[a], Juan Pablo Pinasco[b,*], Cristian Scarola[c]

[a] Firat University, Faculty of Science, Department of Mathematics, Elazig, 23119, Turkey
[b] Departamento de Matemática, FCEyN, Universidad de Buenos Aires, e IMAS, UBA-CONICET, Ciudad Universitaria, Pabellón I (1428) Buenos Aires, Argentina
[c] Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad Nacional de La Pampa, Uruguay 151 (6300), Santa Rosa, La Pampa, Argentina

ARTICLE INFO

Article history:
Received 1 June 2018
Accepted 6 July 2018

Keywords:
Asymptotic behavior
Eigenvalues
Nodal inverse problem

ABSTRACT

In this work we analyze a nonlinear eigenvalue problem for the p-Laplacian operator with zero Dirichlet boundary conditions. We assume that the problem has a potential which depends on the eigenvalue parameter, and we show that, for n big enough, there exists a real eigenvalue λ_n, and their corresponding eigenfunctions have exactly n nodal domains.

We characterize the asymptotic behavior of these eigenvalues, obtaining two terms in the asymptotic expansion of λ_n in powers of n.

Finally, we study the inverse nodal problem in the case of energy dependent potentials, showing that some subset of the zeros of the corresponding eigenfunctions is enough to determine the main term of the potential.

1. Introduction

In this work we consider the following problem

$$-\left(|u'|^{p-2}u'\right)' = [\lambda + g(x,\lambda)]\,|u|^{p-2}u, \qquad x \in (0,1) \tag{1.1}$$

with zero Dirichlet boundary conditions

$$u(0) = u(1) = 0, \tag{1.2}$$

where $1 < p < \infty$, λ is a real parameter, and $g : [0,1] \times [0,+\infty) \to \mathbb{R}$ satisfies the following conditions:

* Corresponding author.
E-mail addresses: hkoyunbakan@gmail.com (H. Koyunbakan), jpinasco@dm.uba.ar (J.P. Pinasco), cristian.scarola@gmail.com (C. Scarola).
URL: http://mate.dm.uba.ar/~jpinasco (J.P. Pinasco).

https://doi.org/10.1016/j.nonrwa.2018.07.001
1468-1218/

(H1) g is a continuous function in $[0,1]\times\mathbb{R}$, and it is locally Lipschitz continuous on the variable λ except perhaps at $\lambda=0$.
(H2) There exists $C>0$, $\alpha\in(0,1)$ such that $|g(x,\lambda)|\le C(\lambda^\alpha+1)$ for any $\lambda\in\mathbb{R}$ and $x\in[0,1]$.

We are interested in the existence of eigenvalues, and these hypotheses are enough to show the existence of an increasing sequence. However, in order to study the asymptotic behavior of the eigenvalues and the inverse nodal problem, we impose an additional condition,

(H3) There exists a continuous function $h(x)$ such that $g(x,\lambda)/\lambda^\alpha\to h(x)$ as $\lambda\to\infty$, uniformly in x.

This kind of problems belong to the class of *nonlinear eigenvalue problems*, where the coefficients of the differential equations depend on the eigenvalue parameter. There exists several particular cases of interest, like the *quadratic eigenvalue problems*, see [1,2]; or the *energy dependent potentials*, see [3–5].

When $p=2$, the eigenvalue problem (1.1)–(1.2) is very important in both classical and quantum mechanics. For example, such problems arise in solving Klein–Gordon equations which describe the motion of massless particles such as photons, they are also used for modeling vibrations of mechanical systems in viscous media, or in hydrodynamic stability problems, see [6]. In particular, generalized one dimensional Schrödinger equations with quadratic operator pencils,

$$-u''=\left(\lambda^2-2\lambda h(x)-q(x)\right)u \tag{1.3}$$

where studied in several papers, see [3,7,8,4,9–12], among other works.

The existence of a discrete set of eigenvalues in the linear case is a difficult problem, and it was solved first by Friedman and Shinbrot [13] using functional analytic techniques for linear compact symmetric operators which can be applied to the inverses of the differential operators. In the one-dimensional case, the work of Greenberg and Babuska [14] uses the shooting method as a way to obtain the eigenvalues; however, this need some monotonicity assumptions on the coefficients. Again in the linear one-dimensional case corresponding to Eq. (1.3), the discreteness of the spectrum and the existence of a double sequence of real eigenvalues going to $\pm\infty$ can be found in [15,16], where additional conditions are imposed on the coefficients, namely,

(G1) $h\in H^1(0,1)$.
(G2) $q\in L^2(0,1)$ satisfies $\int_0^1|u'|^2+q(x)|u|^2dx>0$ for $u\in H^2(0,1)$, $u\neq0$.

Without these conditions, a finite number of complex eigenvalues can appear, see also the work of Browne and Watson [7], where this question is discussed in detail.

Let us remark that this double sequence depend on the particular problem considered. In some cases, Eq. (1.3) appears as

$$-u''=\left(\lambda-2\sqrt{\lambda}h(x)-q(x)\right)u,$$

and only positive values of λ are allowed as eigenvalues. We prefer to work here with this type of problem, since the other case can be handled in much the same way. Let us remark that some pathologies can occur, like the existence of complex eigenvalues, or the existence of two values of λ with the same associated eigenfunctions. We will discuss these questions briefly at the end of Section 2.

The existence of eigenvalues for the p-Laplacian is subtle, since the tools used for $p=2$ are not available in the quasilinear case. Also, a general nonlinearity like $g(x,\lambda)$ forbids the use of the Prüfer transform based on the generalized trigonometric functions (as in [17,18,14]) without additional conditions of monotonicity on g. Nevertheless, we can show that, for sufficiently big n, we have at least one eigenvalue λ_n, and the corresponding eigenfunctions have exactly $n+1$ zeros in $[0,1]$. Our first results in this direction are the following theorems: