Nonlinear Analysis: Real World Applications 45 (2019) 320–356

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

# Global regularity of the 2D magnetohydrodynamic equations with fractional anisotropic dissipation

Zhuan Ye

*Department of Mathematics and Statistics, Jiangsu Normal University, 101 Shanghai Road, Xuzhou 221116, Jiangsu, PR China*

A R T I C L E I N F O

*Article history:*
Received 29 September 2017
Received in revised form 10 July 2018
Accepted 11 July 2018

*Keywords:*
MHD equations
Fractional dissipation
Partial dissipation
Global regularity

A B S T R A C T

This paper is dedicated to establishing the global regularity for the two dimensional magnetohydrodynamic equations with fractional anisotropic dissipation when the fractional powers are restricted to some certain ranges. In addition, the global regularity results for the two dimensional magnetohydrodynamic equations with partial dissipation are also obtained. Consequently, these results bring us more closer to the resolution of the global regularity problem on the two dimensional magnetohydrodynamic equations with standard Laplacian magnetic diffusion.



## 1. Introduction

The magnetohydrodynamic (MHD) equations govern the motion of electrically conducting fluids such as plasmas, liquid metals, and electrolytes. They can be obtained through a coupling of incompressible Navier–Stokes and Maxwell's equations and play a fundamental role in geophysics, astrophysics, cosmology and engineering (see, e.g., [1]). For more physical background, one may check the Refs. [2,3] for more detailed explanations. Because of its profound physical background and important mathematical significance, there is a growing literature devoted to the MHD equations by many physicists and mathematicians so that it is almost impossible to be exhaustive in this introduction. Here we only recall the notable works about the regularity results of the two-dimensional (2D) incompressible MHD equations (see [4–14]. However, the dynamic motion of the fluid and the magnetic field interact on each other and both the hydrodynamic and electrodynamic effects in the motion are strongly coupled, the MHD equations are more sophisticated. Therefore, numerous outstanding open problems in applied analysis have not been settled completely. For example, the global regularity for the 2D MHD equations with the standard Laplacian velocity dissipation or with the standard Laplacian magnetic diffusion remains completely open.

*E-mail address:* yezhuan815@126.com.

https://doi.org/10.1016/j.nonrwa.2018.07.014
1468-1218/

It is well-known that the 2D MHD equations with both Laplacian dissipation and magnetic diffusion have a unique global classical solution (see [15]). However, on the other hand, the regularity of the completely inviscid MHD equations appears to be out of reach. Comparing these the two extreme cases, it is natural for us to consider the intermediate cases, i.e., the 2D MHD equations with partial dissipation, and to ask of whether the solutions exist globally in time or blow up in finite time for these intermediate cases. Recently, this direction of research has caught much attention from many mathematicians and a remarkable development with new results has been seen (see [6,7,16,8,17,11,18–23] for fractional dissipation and see [4,5,24,9,25,10,26,12–14] for partial dissipation). Moreover, the global existence of the classical solution when the initial data close to the equilibrium state was proved by many works recently, see for example [27,9,10,28,26,12,13,29,30,14]. At this point it is worth mentioning that in certain physical regimes and under suitable scaling, the full Laplacian dissipation can be reduced to a partial dissipation. One remarkable example is the Prandtl boundary layer equation in which only the vertical dissipation is included in the horizontal component, see [31] for details. Starting with this observation, a natural question arises: what if we do not impose the full Laplacian dissipation? To get the global well-posedness, is it enough to keep some components of the standard fractional Laplacian while drop the others? In other words, we are interested in the structure of the anisotropic hyperdissipative (anisotropic fractional Laplacian) operator. More precisely, we are interested in studying the 2D MHD equations with fractional vertical dissipation in the horizontal velocity, fractional horizontal dissipation in the vertical velocity and fractional vertical dissipation in the horizontal magnetic field, fractional horizontal dissipation in the vertical magnetic field. More precisely, it takes the following form

$$\begin{cases} \partial_t u_1 + (u\cdot\nabla)u_1 + \Lambda_{x_2}^{2\alpha}u_1 + \partial_{x_1}p = (b\cdot\nabla)b_1, \\ \partial_t u_2 + (u\cdot\nabla)u_2 + \Lambda_{x_1}^{2\alpha}u_2 + \partial_{x_2}p = (b\cdot\nabla)b_2, \\ \partial_t b_1 + (u\cdot\nabla)b_1 + \Lambda_{x_2}^{2\beta}b_1 = (b\cdot\nabla)u_1, \\ \partial_t b_2 + (u\cdot\nabla)b_2 + \Lambda_{x_1}^{2\beta}b_2 = (b\cdot\nabla)u_2, \\ \nabla\cdot u = \nabla\cdot b = 0, \end{cases} \tag{1.1}$$

subjected with initial data

$$u(x,0) = u_0(x), \qquad b(x,0) = b_0(x),$$

where $u = u(x,t) = (u_1(x,t), u_2(x,t))$ denotes the velocity field, $p = p(x,t)$ denotes the pressure, $b = b(x,t) = (b_1(x,t), b_2(x,t))$ denotes the magnetic field at point $(x_1, x_2) \in \mathbb{R}^2$ at time $t > 0$, respectively. The fractional horizontal operator $\Lambda_{x_1} := \sqrt{-\partial_{x_1}^2}$ and the fractional vertical operator $\Lambda_{x_2} := \sqrt{-\partial_{x_2}^2}$ are defined through the Fourier transform, namely

$$\widehat{\Lambda_{x_1}^{\sigma} f}(\xi) = |\xi_1|^{\sigma}\hat{f}(\xi), \qquad \widehat{\Lambda_{x_2}^{\sigma} f}(\xi) = |\xi_2|^{\sigma}\hat{f}(\xi), \qquad \xi = (\xi_1, \xi_2),$$

where

$$\hat{f}(\xi) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{-ix\cdot\xi} f(x)\,dx.$$

We make the convention that by $\alpha = 0$ we mean that there is no dissipation in the velocity equation, and similarly $\beta = 0$ means that there is no dissipation in the magnetic field. We would like to point out that by establishing an anisotropic product inequality, Cao and Wu [5] established the global regularity of the classical solutions for the 2D MHD equations with mixed partial dissipation and magnetic diffusion. Inspired by this elegant work, Du and Zhou [24] considered 34 different cases and provided us with many global regularity results as well as sufficient conditions in the cases that remain very challenging open problems. In particular, Du and Zhou [24] proved the global existence of classical solutions for the system (1.1) with $\alpha = \beta = 1$. The main purpose of this paper is to establish the global well-posedness of classical solutions of the system (1.1) to improve these results. The main result of the present paper is stated in the following theorem.