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Averaging methods for nonlinear systems with a small parameter via reduction and topological degree

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HIGHLIGHTS

- We study the continuation of periodic solutions with non-simple zeros.
- Averaging is constructed by the Lyapunov–Schmidt reduction and topological theory.
- Periodic solutions of the forced Galactic dynamics model are obtained.

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ABSTRACT

When using averaging theory, one usually tries to find the simple zeros of an associated averaged function. When the zero is not simple, the classical averaging theory does not provide information about the periodic solution associated to a non-simple zero. To overcome this difficulty, one way is to perform the high order averaging in case that the system is smooth enough. Here we provide sufficient conditions to keep periodic solutions persisting by the Lyapunov–Schmidt reduction and topological degree theory. Additionally, we perform an application of our result for the forced Galactic dynamics model.

1. Introduction

The method of averaging is an important tool for studying the time-periodic nonlinear differential systems with small parameters, which has been applied in various research fields such as celestial mechanics [1,2], condensed matter physics [3,4], quantum calculus [5]. The first formalization of this theory was established by Fatou [6], and we can refer to the book Sanders–Verhulst–Murdock [7] for a modern exposition. The method of averaging has been improved in various directions. For example, see [8–13] for a new development.

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Consider the bifurcation of T-periodic solutions for the time T-periodic nonlinear differential system with a small parameter

$$x'(t) = F_0(t,x) + \varepsilon F_1(t,x) + \varepsilon^2 R(t,x,\varepsilon), \tag{1.1}$$

where ε is a small parameter, $F_0, F_1 : \mathbb{R} \times \bar{\Omega} \to \mathbb{R}^n$ and $R : \mathbb{R} \times \bar{\Omega} \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are C^2 functions, T-periodic in the first variable, and Ω is an open subset of $\mathbb{R}^n$. One of the main hypotheses is that the unperturbed system

$$x'(t) = F_0(t,x) \tag{1.2}$$

has a manifold of periodic solutions. This problem was solved earlier by Malkin (1956) and Roseau (1966), and a new and concise proof was given by Buică, Françoise and Llibre [8, Theorem 3.1] where they combined the method of averaging with the theory of Lyapunov–Schmidt reduction.

Very inspired by the recent work [14], we study the problem of the bifurcation of T-periodic solutions for system (1.1) with the degenerate Jacobi matrix, which leads to that the direct application of the implicit function theorem is unavailable. Our ideas motivated by [8,14], and we use the method of averaging together with topological degree theory and the Lyapunov–Schmidt reduction. In [8], the implicit function theorem has been used twice. When the Jacobi matrix is degenerate, in order to apply the implicit function theorem one has to use the high order averaging, see [15] for instance. However, to reduce computational complexity, in this paper we use the topological degree theory to solve the bifurcation equation.

The paper is organized as follows. In Section 2, we introduce the theory of Lyapunov–Schmidt reduction for finite dimensional functions, and a result on solutions of algebraic equation (see Theorem 2.1) is obtained by the method of Lyapunov–Schmidt reduction and the theory of topological degree, which will be used later in the proof of Theorem 3.1 in Section 3. Theorem 3.1 extends the result ([8, Theorem 3.1]) to the degenerate case and the proof of Theorem 3.1 follows the method of Buică, Françoise and Llibre, see [8]. In the final, we give an application of Theorem 3.1 for the forced Galactic dynamics model. The existence of the periodic solutions has been established in various cases, see Theorems 4.1 and 4.2.

2. Lyapunov–Schmidt reduction

In this section, we will introduce a result about the Lyapunov–Schmidt reduction for finite dimensional functions. It is a special case of the general theory [16], and the proof is simple involving the theory of topological degree. The theorem stated below will be used later in the proof of the averaging theorem in the next section. We mention that the function $F(\alpha)$ that appears in the proof of the following theorem is called the bifurcation function.

Theorem 2.1. *Assume that*

$$f = (f_1, f_2) : \mathbb{R}^k \times \mathbb{R}^{n-k} \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^k \times \mathbb{R}^{n-k},$$
$$(x, y, \varepsilon) \mapsto (f_1(x,y,\varepsilon), f_2(x,y,\varepsilon))$$

is a C^2 smooth function, $\Omega \subseteq \mathbb{R}^k$ is an open set and the function $\beta_0 : \bar{\Omega} \to \mathbb{R}^{n-k}$ is of class C^2. Moreover, the following conditions hold.

(i) for all $\alpha \in \bar{\Omega}$, we have $f(\alpha, \beta_0(\alpha), 0) = 0$.

(ii) The Jacobi matrix has the form

$$\left.\frac{\partial(f_1, f_2)}{\partial(x,\ y)}\right|_{\substack{x=\alpha,\, \varepsilon=0,\\ y=\beta_0(\alpha)}} = \begin{pmatrix} \partial f_1/\partial x & 0 \\ \partial f_2/\partial x & \partial f_2/\partial y \end{pmatrix} \tag{2.1}$$