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Periodic solutions for Liénard equation with an indefinite singularity☆

Shiping Lu*, Yuanzhi Guo, Lijuan Chen

College of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing, 210044, PR China

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A B S T R A C T

In this paper, the problem of periodic solutions is studied for Liénard equations with an indefinite singularity

$$x''(t) + f(x(t))x'(t) + \varphi(t)x^m(t) - \frac{\alpha(t)}{x^\mu(t)} = 0,$$

where $f : (0, +\infty) \to \mathbb{R}$ is a continuous function which may have a singularity at the origin, the signs of φ andα are allowed to change, m is a non-negative constant, and μ is a positive constant. The approach is based on a continuation theorem of Manásevich and Mawhin with techniques of a priori estimates. The main results partly answer the open problem proposed by R. Hakl, P.J. Torres and M. Zamora in the known literature.

1. Introduction

In this paper, we are concerned with the existence of positive T-periodic solutions for the equations with an indefinite singularity

$$x''(t) + f(x(t))x'(t) + \varphi(t)x^m(t) - \frac{\alpha(t)}{x^\mu(t)} = 0, \tag{1.1}$$

where $f \in C((0, +\infty), \mathbb{R})$, φ and α are T-periodic functions with φ and α in $L([0, T], R)$, m, μ are constants with $m \geq 0$ and $\mu > 0$. In this equation, the function $f(x)$ may have a singularity at $x = 0$. Besides this, the signs of $\alpha(t)$ and $\varphi(t)$ are all allowed to change. The equations of this type arise in modeling of important problems appearing in many physical contexts (see [1–6] and the references therein).

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* Corresponding author.
E-mail addresses: lushiping88@sohu.com (S. Lu), 1029158296@qq.com (Y. Guo), 002084@nuist.edu.cn (L. Chen).

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In the past years, under the conditions of $\varphi(t) \geq 0$ and $\alpha(t) > 0$ for a.e. $t \in [0, T]$, the problem of existence of periodic solutions to the equation without friction term

$$x''(t) + \varphi(t)x(t) - \frac{\alpha(t)}{x^{\mu}(t)} = 0$$

has been extensively studied [7–21]. Beginning with the paper of Habets–Sanchez [22], many researchers in [23–30] have considered the classical Liénard equation with a singularity of repulsive type

$$x''(t) + f(x(t))x'(t) + \varphi(t)x(t) - \frac{l}{x^{\mu}(t)} = 0, \tag{1.2}$$

where $l > 0$ is a constant. In these papers, apart from the function $\varphi(t)$ satisfies $\varphi(t) \geq 0$ for a.e. $t \in [0, T]$, the strong force condition $\mu \in [1, +\infty)$ and $f(x)$ being continuous on $[0, +\infty)$ are needed. But up to our knowledge, few papers have considered the possibility of a singularity in $f(x)$. We only find that R. Hakl, PJ. Torres and M. Zamora in [27] and [31] studied the problem of periodic solutions for the singular equation of repulsive type

$$x'' + f(x)x' + \varphi(t)x^{m} + \frac{g_1}{x^{\mu}} - \frac{g_2}{x^{\gamma}} = 0, \tag{1.3}$$

where g_1 and g_2 are constants with $g_1 \geq 0$ and $g_2 > 0$, $\varphi \in L([0, T], \mathbb{R})$ is a T-periodic function with $\bar{\varphi} \geq 0$, and $f \in C((0, +\infty), \mathbb{R})$ may have a singularity at $x = 0$, i.e., $\lim_{x \to 0^+} f(x) = \infty$. By using Schaefer's fixed point theorem and the method of upper and lower solutions, they obtained many results on the existence of positive periodic solutions to (1.3) for the case of $0 \leq m < +\infty$ and $\mu < \gamma$. Now, we present some of them.

Theorem 1.1 *([27]). Let us assume $0 \leq m < 1$, $\gamma > \mu$, $\gamma \geq 1$, and*

$$\int_0^1 \max\{f(s), 0\} ds < +\infty \text{ or } \int_0^1 \min\{f(s), 0\} ds > -\infty. \tag{1.4}$$

Then (1.3) has at least one positive T-periodic solution.

Theorem 1.2 *([27]). Let us assume $m = 1$, $\gamma > \mu$, $\gamma \geq 1$, and suppose that (1.4) holds. If $\bar{\varphi} \geq 0$, $g_1 + |\bar{\varphi}| > 0$ and*

$$\int_0^T \max\{\varphi(s), 0\} ds < \frac{4}{T},$$

then (1.3) has at least one positive T-periodic solution.

Theorem 1.3 *([31]). Let us assume $m > 1$, $\gamma > \mu$, $g_1 > 0$ and $g_2 > 0$. If*

$$0 \geq \varphi(t) \geq -\sup\left\{\frac{g_1}{x^{\mu}} - \frac{g_2}{x^{\gamma}} : x \in (0, +\infty)\right\} \text{ for } a.e. t \in [0, \omega] \tag{1.5}$$

and $\bar{\varphi} < 0$, then (1.3) has at least one positive T-periodic solution.

From Theorems 1.1 and 1.2, we see that the constant m in (1.3), which is the degree of power function $\varphi(t)x^{m}$, is needed in $[0, 1]$. Furthermore, the constants of γ and μ in (1.3) are required to be $\gamma > \mu$ and $\gamma \geq 1$, which implies that the singular restoring force term $(\frac{g_1}{x^{\mu}} - \frac{g_2}{x^{\gamma}})$ satisfies the strong force condition, i.e.,

$$\int_0^1 \left(\frac{g_1}{s^{\mu}} - \frac{g_2}{s^{\gamma}}\right) ds = -\infty.$$