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Existence and multiplicity of solutions for Kirchhoff–Schrödinger type equations involving $p(x)$-Laplacian on the entire space $\mathbb{R}^N$

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ABSTRACT

This study is concerned with the following elliptic equation:

$$-M\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)+V(x)|u|^{p(x)-2}u = \lambda f(x,u) \quad \text{in } \mathbb{R}^N,$$

where $M \in C(\mathbb{R}^+)$ is a Kirchhoff-type function, the potential function $V : \mathbb{R}^N \to (0,\infty)$ is continuous, and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies a Carathéodory condition. The aim is to determine the precise positive interval of λ for which the problem admits at least two nontrivial solutions by using abstract critical point results for an energy functional satisfying the Cerami condition. It should be noted that the existence of at least one nontrivial weak solution is established by employing the mountain pass theorem. Moreover, the existence of an unbounded sequence of nontrivial weak solutions follows from the fountain theorem owing to the variational nature of the problem.

1. Introduction

Recently, the study of variational problems with nonstandard growth conditions in, e.g., elastic mechanics, electro-rheological fluid dynamics, and image processing, *etc*, has attracted a great deal of attention; see [1–4].

The present study is concerned with the existence and multiplicity of nontrivial weak solutions for Kirchhoff-type equations involving the $p(x)$-Laplacian, namely,

$$-M\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)+V(x)|u|^{p(x)-2}u=\lambda f(x,u) \quad \text{in } \mathbb{R}^N, \tag{P}$$

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where $M \in C(\mathbb{R}^+)$ is a Kirchhoff-type function, the potential function $V : \mathbb{R}^N \to (0,\infty)$ is continuous, and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies a Carathéodory condition. Problems involving the $p(x)$-Laplacian have been extensively studied [5–7]. Moreover, Kirchhoff in [8] initially proposed the following equation:

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right| dx\Big)\frac{\partial^2 u}{\partial x^2} = 0,$$

which is a generalization of the classical D'Alembert's wave equation. Furthermore, Woinowsky and Krieger [9] in the 1950s, considered the following evolution equation of Kirchhoff type:

$$u_{tt} + \Delta^2 u - M(\|\nabla u\|^2)\Delta u = f(x,u),$$

which is a model for the deflection of an extensible beam. Here, u denotes the displacement, f is the force, and M models the effect of small changes in the length of the beam (see, e.g., [10–13] for the physics background and the related models). From a mathematical perspective, the existence of weak solutions for elliptic problems of Kirchhoff type has been extensively studied [14–20].

Since the pioneering work of A. Ambrosetti and P. Rabinowitz in [21], critical point theory has become a major technique for determining solutions to elliptic equations of variational type [6,22–24]. In particular, existence and multiplicity results for the $p(x)$-Laplacian Dirichlet problems were obtained by X. Fan and Q.H. Zhang [6]; the case of the entire domain $\mathbb{R}^N$ was treated in [5]. The key ingredient for obtaining these results is the condition by A. Ambrosetti and P. Rabinowitz ((AR)-condition for short).

(AR) There exist positive constants m and ζ such that $\zeta > p_+$ and

$$0 < \zeta F(x,t) \le f(x,t)t \quad \text{for} \quad x \in \Omega \quad \text{and} \quad |t| \ge m,$$

where $p_+ = \sup_{x\in\Omega} p(x)$, $F(x,t) = \int_0^t f(x,s)\,ds$, and Ω is a bounded domain in $\mathbb{R}^N$.

It is well known that the (AR)-condition is natural and important in ensuring the boundedness of the Palais–Smale sequence of the Euler–Lagrange functional, which plays a decisive role in applying critical point theory. However, this condition is quite restrictive and eliminates several nonlinearities. Thus, it has been recently attempted to drop it [25–31].

In particular, C.O. Alves and S.B. Liu [32] obtained results similar to those in [33] for the case of the entire space $\mathbb{R}^N$. To verify them, the following condition was assumed:

(Je) There exists $\eta \ge 1$ such that

$$\eta\mathcal{F}(x,t) \ge \mathcal{F}(x,st)$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ and $s \in [0,1]$, where $\mathcal{F}(x,t) = f(x,t)t - p_+F(x,t)$.

Condition (Je) originates in the study of L. Jeanjean [34] in the case $p(x) \equiv 2$ (see also [32]) and is weaker than certain assumptions given in [33]. Recently, a great deal of research has been conducted on the p-Laplacian problem under assumption (Je) [27,28]; see also [31] for the case of variable exponents $p(x)$. X. Lin and X.H. Tang [35] established the existence of a nontrivial weak solution of p-Laplacian equations under mild assumptions for the nonlinear growth f that are substantially different from those studied in [28,29,34,35]. The main point in these studies is to establish the existence of weak solutions to certain elliptic equations applying the mountain pass theorem or the fountain theorem under a compactness condition for an energy functional.