Nonlinear Analysis: Real World Applications 45 (2019) 650–667

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

Limit cycles of polynomial Liénard systems via the averaging method

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ARTICLE INFO

Article history:
Received 3 January 2017
Received in revised form 23 July 2018
Accepted 25 July 2018

Keywords:
Polynomial Liénard system
Limit cycle bifurcation
Averaging method

ABSTRACT

In this paper we provide an upper bound on the maximum number of limit cycles for a class of generalized polynomial Liénard differential systems $\dot{x} = y$, $\dot{y} = -f_n(x)y - g_m(x)$ with f_n and g_m real polynomials of degree n and m respectively, using the fourth order averaging method.

1. Introduction and statement of the main results

For polynomial Liénard differential equations

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{1}$$

with $f(x)$ and $g(x)$ polynomials of degree n and m respectively, there appeared a conjecture on the maximum number of limit cycles of system (1) with $g(x) = x$, posed by Lins Neto, de Melo and Pugh [1] in 1977, which states that system (1) with $g(x) = x$ has at most $\frac{n-1}{2}$ limit cycles. This conjecture was verified only for $n \leq 4$, where for $n = 1, 2, 3$ it was proved by Lins Neto et al. [1], and for $n = 4$ by Li and Llibre [2]. For $n > 5$ the conjecture is not correct, see [3–5]. De Maesschalck and Huzak proved in 2015 that the classical Liénard system could have $n - 2$ limit cycles for $n \geq 6$.

For applying the averaging methods to study the number of limit cycles of system (1), one considers system (1) with

$$\begin{aligned} f(x) &= \varepsilon f_1(x) + \varepsilon^2 f_2(x) + \varepsilon^3 f_3(x) + \varepsilon^4 f_4(x), \\ g(x) &= x + \varepsilon g_1(x) + \varepsilon^2 g_2(x) + \varepsilon^3 g_3(x) + \varepsilon^4 g_4(x), \end{aligned} \tag{2}$$

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https://doi.org/10.1016/j.nonrwa.2018.07.030
1468-1218/

where f_i and g_i are polynomials of degree n and m for $i = 1, 2, 3, 4$. Equivalently, we write Eq. (1) in the Liénard differential systems

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - \left(\varepsilon f_1(x) + \varepsilon^2 f_2(x) + \varepsilon^3 f_3(x) + \varepsilon^4 f_4(x)\right) y \\ &\qquad - \left(\varepsilon g_1(x) + \varepsilon^2 g_2(x) + \varepsilon^3 g_3(x) + \varepsilon^4 g_4(x)\right). \end{aligned} \tag{3}$$

In 2010 Llibre, Mereu and Teixeira [6] obtained the maximum number of limit cycles of the polynomial system (3) with f, g given in (2) bifurcating from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$ through the third order averaging method. The maximum number is

(a) $\left[\frac{n}{2}\right]$, via the first order averaging method;
(b) $\max\left\{\left[\frac{n-1}{2}\right] + \left[\frac{m}{2}\right], \left[\frac{n}{2}\right]\right\}$ via the second order averaging method;
(c) $\left[\frac{n+m-1}{2}\right]$ via the third order averaging method;

Here we will continue this work to provide an upper bound on the number of limit cycles of system (3) using the fourth order averaging method.

Our main result is the following.

Theorem 1. *Assume that f_i and g_i for $i = 1, \ldots, 4$ are polynomials of degree n and m respectively. Then for $|\varepsilon| > 0$ sufficiently small, the maximum number, say $H_4(n, m)$, of limit cycles of the polynomial Liénard differential systems (3) bifurcating from the periodic orbits of the linear center $\dot{x} = y, \dot{y} = -x$ using the fourth order averaging theory is no more than*

$$H_4(m,n) = \begin{cases} 4n+1, & m < n+1, n \text{ odd}, \\ 4m-3, & m \geq n+1, n \text{ odd}, \\ 4n-3, & m < n+1, n \text{ even}, \\ 4m-3, & m \geq n+1, n \text{ even}. \end{cases}$$

At the moment we do not know if the upper bound in Theorem 1 can be reached.

The rest of this paper is contributed to the proof of Theorem 1, where lots of detailed calculations were omitted because of their big blocks. If readers are interested in them we would like to provide them in an additional sheet.

2. Proof of Theorem 1

The mail tool of our proof to Theorem 1 is the fourth order averaging method. For readers' convenience and in order that this paper is self-contained, we recall this method here.

2.1. The fourth order averaging method

The following result provides the first four averaging functions of a periodic differential equation. For more details, see [7].

Lemma 1 (*The Fourth Order Averaging Method*)**.** *Consider the following first order periodic differential equation of period T*

$$\dot{x} = \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x) + \varepsilon^3 F_3(t,x) + \varepsilon^4 F_4(t,x) + \varepsilon^5 R(t,x,\varepsilon), \tag{4}$$

where $F_i \in C^0(\mathbb{R} \times D, \mathbb{R})$ and $R \in C^0(\mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0), \mathbb{R})$ are periodic functions of period T in t for $i = 1, 2, 3, 4$ and $D \subset \mathbb{R}$ an open set. We have the next assumptions.