Nonlinear Analysis: Real World Applications 45 (2019) 668–687

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

Stability estimates for non-local scalar conservation laws

Felisia Angela Chiarello, Paola Goatin*, Elena Rossi

Inria Sophia Antipolis - Méditerranée, Université Côte d'Azur, Inria, CNRS, LJAD, 2004 route des Lucioles - BP 93, 06902 Sophia Antipolis Cedex, France

ARTICLE INFO

Article history:
Received 15 January 2018
Received in revised form 22 July 2018
Accepted 23 July 2018

Keywords:
Scalar conservation laws
Non-local flux
Stability

ABSTRACT

We prove the stability of entropy weak solutions of a class of scalar conservation laws with non-local flux arising in traffic modelling. We obtain an estimate of the dependence of the solution with respect to the kernel function, the speed and the initial datum. Stability is obtained from the entropy condition through doubling of variable technique. We finally provide some numerical simulations illustrating the dependencies above for some cost functionals derived from traffic flow applications.

1. Introduction

Conservation laws with non-local flux have drawn growing attention in the recent years. Indeed, beside the intrinsic mathematical interest for their properties, they turned out to be suitable for modelling several phenomena arising in natural or engineering sciences: flux functions depending on space-integrals of the unknown appear for example in models for granular flows [1], sedimentation [2], supply chains [3], conveyor belts [4], weakly coupled oscillators [5], structured populations dynamics [6] and traffic flows [7–9].

For this type of equations, general existence and uniqueness results have been established in [10,11] for specific classes of scalar equations in one space-dimension, and in [12] for multi-dimensional systems of equations coupled through the non-local term. In particular, existence is usually proved by providing suitable compactness estimates on a sequence of approximate solutions constructed by finite volume schemes, while $\mathbf{L^1}$-stability on initial data is obtained from Kružkov-type entropy conditions through the doubling of variable technique [13]. A different approach based on fixed-point techniques has been recently proposed in [14] to prove existence and uniqueness of solutions to scalar balance laws in one space dimension, whose velocity term depends on the weighted integral of the density over an area in space.

* Corresponding author.
E-mail addresses: felisia.chiarello@inria.fr (F.A. Chiarello), paola.goatin@inria.fr (P. Goatin), elena.rossi@inria.fr (E. Rossi).

https://doi.org/10.1016/j.nonrwa.2018.07.027
1468-1218/

In this paper, we focus on a specific class of scalar equations, in which the integral dependence of the flux function is expressed though a convolution product. We consider the following Cauchy problem

$$\begin{cases} \partial_t \rho + \partial_x \left(f(t,x,\rho) V(t,x) \right) = 0 & t > 0,\ x \in \mathbb{R}, \\ \rho(0,x) = \rho_o(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $V(t,x) = v\left((\rho(t) * w)(x)\right)$, and w is a smooth mollifier:

$$(\rho(t) * w)(x) = \int_{\mathbb{R}} \rho(t,y)\, w(x-y) \mathrm{d}y.$$

Here and below, we set $\rho(t) := \rho(t,\cdot)$ the function $x \mapsto \rho(t,x)$.

Existence and uniqueness of solutions to (1.1) follows from [10], as well as some *a priori* estimates, namely $\mathbf{L^1}$, $\mathbf{L^\infty}$ and total variation estimates, see Section 2.

Motivated by the study of control and optimisation problems with focus on traffic management, we are interested in studying the dependence of solutions to (1.1) on the convolution kernel w and on the velocity function v. While controlling the speed may seem more straightforward, varying the interaction kernel could be of interest in applications to connected autonomous vehicles.

Estimates of the dependence of solutions to a general balance law on the flux function can be found in [15,16]. However, as precised also below (see Remark 3), those estimates turn out to be implicit when applied to the setting of problem (1.1).

Carefully applying the Kružkov's doubling of variables techniques, on the lines of [2,17], we derive the $\mathbf{L^1}$-Lipschitz continuous dependence of solutions to (1.1) on the initial datum, the kernel (see Theorem 1) and the velocity (see Theorem 2). These results are collected in Section 2, while the technical proofs are deferred to Section 3. Finally, in Section 4 we show some numerical simulation illustrating the behaviour of the solutions of a non-local traffic flow model, when the size and the position of the kernel support or the velocity function vary. In particular, we analyse the impact on two cost functionals, measuring traffic congestion.

2. Main results

The study of problem (1.1) is carried out in the same setting of [10], with slightly strengthened conditions. We recall here briefly the assumptions on the flux function f, on v and on w:

$$f \in \mathbf{C^2}(\mathbb{R} \times \mathbb{R} \times \mathbb{R}; \mathbb{R}^+) \quad \text{and} \quad \begin{cases} \sup_{t,x,\rho} \left| \partial_\rho f(t,x,\rho) \right| & < & +\infty \\ \sup_{t,x} \left| \partial_x f(t,x,\rho) \right| & < & C\left|\rho\right| \\ \sup_{t,x} \left| \partial^2_{xx} f(t,x,\rho) \right| & < & C\left|\rho\right| \\ \forall t,\ x \quad f(t,x,0) & = & 0 \end{cases} \tag{2.1}$$

$$v \in (\mathbf{C^2} \cap \mathbf{W}^{2,\infty})(\mathbb{R};\mathbb{R}) \quad \text{and} \quad w \in (\mathbf{C^2} \cap \mathbf{W}^{1,1} \cap \mathbf{W}^{2,\infty})(\mathbb{R};\mathbb{R}). \tag{2.2}$$

Throughout the paper, we make use of the following definition of solution to problem (1.1), see also [10, Definition 2.1].

Definition 1. Let $T > 0$. Fix $\rho_o \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$. A *weak entropy solution* to (1.1) on $[0,T]$ is a bounded measurable Kružkov solution $\rho \in \mathbf{C^0}([0,T]; \mathbf{L^1_{loc}}(\mathbb{R};\mathbb{R}))$ to

$$\begin{cases} \partial_t \rho + \partial_x \left(f(t,x,\rho)\, V(t,x) \right) = 0 \\ \rho(0,x) = \rho_o(x) \end{cases} \quad \text{where} \quad V(t,x) = v((\rho(t) * w)(x)),$$