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Stability and multiplicity of standing waves for the inhomogeneous NLS equation with a harmonic potential[☆]

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ABSTRACT

In this article, we study the stability and multiplicity of normalized standing waves for the inhomogeneous nonlinear Schrödinger equation with a harmonic potential arising in nonlinear optics.

1. Introduction and main results

In this paper, we study the stability and multiplicity of standing waves with a prescribed L^2-norm for the following inhomogeneous nonlinear Schrödinger equation

$$\begin{cases} iu_t + \Delta u - |x|^2 u + |x|^b |u|^{p-2} u = 0, \ (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where t denotes the time, $N \geq 2$ is the space dimension, $b > 0$ and $\underline{p} := 2 + \frac{2b}{N-1} < p < \overline{p}$ with

$$\overline{p} = \begin{cases} \dfrac{2N}{N-2} + \dfrac{2b}{N-1}, & \text{if } N \geq 3 \\ +\infty, & \text{if } N = 2. \end{cases}$$

Problem (1.1) arises in nonlinear optics. The nonlinearity is usually considered in the form of $f(x,|u|^2)u$ (see [1–3]), where $f(x,|u|^2)$ is the nonlinear index of refraction and depends on the medium. Berge in [1]

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formally studied the stability condition for soliton solutions depending on the shape of $f(x,|u|^2)$. In this paper, we assume that the nonlinear index of refraction $f(x,|u|^2) = |x|^b|u|^{p-2}$ with $b > 0$ and study the stable standing waves for problem (1.1). For the case $b < 0$, we refer the reader to [2–7].

We recall some previous results on problem (1.1). In [8,9], by establishing an improved inequality of Gagliardo–Nirenberg type interpolation: for $N \geq 2$, $b \geq 0$ and $p \in (\underline{p}, \bar{p})$, there exists a positive constant $C(N,p,b)$ such that for any $u \in H_r^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} |x|^b|u|^p \leq C(N,p,b)\big(\int_{\mathbb{R}^N} |\nabla u|^2\big)^{\frac{N(p-2)-2b}{4}}\big(\int_{\mathbb{R}^N} |u|^2\big)^{\frac{2p+2b-N(p-2)}{4}}, \tag{1.2}$$

the authors get a sharp criterion for the global existence and blow up of solutions to problem (1.1). In [10], the authors proved the instability of standing waves for problem (1.1) under suitable conditions. As far as we know, there are no results concerning the stability of standing waves for problem (1.1).

When the problem is free of the electromagnetic trap, i.e. (1.1) without the term $|x|^2u$, we refer the reader to [11–13] and the references therein for more information.

Physicists are often interested in "normalized solutions" to problem (1.1), i.e. solutions to problem (1.1) satisfying

$$\int_{\mathbb{R}^N} |u(t,x)|^2dx = c, \tag{1.3}$$

where $c > 0$ is a fixed constant and independent on t. The motivation is that the mass

$$\int_{\mathbb{R}^N} |u(t,x)|^2dx$$

is preserved along the trajectories of (1.1), see ([14], Proposition 3.1). In order to find standing waves, we set in (1.1)

$$u(t,x) = e^{i\omega t}v(x),\ x \in \mathbb{R}^N,\ t > 0,$$

then $v : \mathbb{R}^N \to \mathbb{R}$ satisfies the following stationary equation

$$-\Delta v + |x|^2v + \omega v - |x|^b|v|^{p-2}v = 0, x \in \mathbb{R}^N \tag{1.4}$$

with

$$\int_{\mathbb{R}^N} |v(x)|^2dx = c.$$

Throughout this paper, we denote the norm of $L^p(\mathbb{R}^N)$ by

$$\|u\|_p := \big(\int_{\mathbb{R}^N} |u|^p\big)^{\frac{1}{p}}$$

for any $1 \leq p < \infty$. The Hilbert space Σ is defined as

$$\Sigma := \Big\{u \in H^1(\mathbb{R}^N) : u(x) = u(|x|), \int_{\mathbb{R}^N} |x|^2|u|^2 < +\infty\Big\},$$

with the inner product and norm

$$(u,v)_\Sigma := \int_{\mathbb{R}^N} (\nabla u\nabla v + |x|^2uv + uv)dx,\ \|u\|_\Sigma := (\|\nabla u\|_2^2 + \|xu\|_2^2 + \|u\|_2^2)^{\frac{1}{2}}.$$

We use "$\to$" and "$\rightharpoonup$" respectively to denote the strong and weak convergence in the related function spaces. C will denote a positive constant unless specified.