Nonlinear Analysis: Real World Applications 45 (2019) 704–720

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

Structure of the set of stationary solutions to the equations of motion of a class of generalized Newtonian fluids

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ARTICLE INFO

Article history:
Received 26 February 2018
Received in revised form 19 July 2018
Accepted 25 July 2018

Keywords:
Generalized Newtonian fluid
Equations of motion
Stationary solutions

ABSTRACT

We investigate the steady-state equations of motion of the generalized Newtonian fluid in a bounded domain $\Omega \subset \mathbb{R}^N$, when $N = 2$ or $N = 3$. Applying the tools of nonlinear analysis (Smale's theorem, theory of Fredholm operators, etc.), we show that if the dynamic stress tensor has the 2-structure then the solution set is finite and the solutions are C^1-functions of the external volume force $\mathbf{f}$ for generic $\mathbf{f}$. We also derive a series of properties of related operators in the case of a more general p-structure, show that the solution set is compact if $p > 3N/(N+2)$ and explain why the same approach as in the case $p = 2$ cannot be applied if $p \neq 2$.

1. Introduction

1.1. The dynamic stress tensor

We denote by $\mathbf{v}$ the velocity of a moving fluid and by $\mathbb{D}\mathbf{v}$ the rate of deformation tensor, which coincides with the symmetric gradient of velocity: $\mathbb{D}\mathbf{v} := \frac{1}{2}[\nabla\mathbf{v} + (\nabla\mathbf{v})^T]$. It is well known that the dynamic stress tensor $\mathbb{S}$ in the so called Stokesian fluid generally depends on the rate of deformation tensor through the formula

$$\mathbb{S}(\mathbb{D}\mathbf{v}) \;=\; \alpha\,\mathbb{I} + \beta\,\mathbb{D}\mathbf{v} + \gamma\,(\mathbb{D}\mathbf{v})^2, \tag{1.1}$$

where the coefficients α, β and γ may further depend on the state variables (pressure q, density ρ, temperature ϑ) and on the principal invariants of tensor $\mathbb{D}$. (See e.g. [1, Sec. 5.21, 5.22].) If the density is constant (which implies that the fluid is incompressible and the first principal invariant of $\mathbb{D}\mathbf{v}$, i.e. $\operatorname{div}\mathbf{v}$, is equal to zero)

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https://doi.org/10.1016/j.nonrwa.2018.07.029
1468-1218/

and the fluid is Newtonian (which means that the dependence of $\mathbb{S}$ on $\mathbb{D}\mathbf{v}$ is linear) then (1.1) reduces to the form $\mathbb{S}(\mathbb{D}\mathbf{v}) = \beta\,\mathbb{D}\mathbf{v}$, where β still may depend on q and ϑ. Considering particularly $\beta = 2\mu$, where μ is the so called dynamic coefficient of viscosity, and substitution of $\mathbb{S}(\mathbb{D}\mathbf{v}) = 2\mu\,\mathbb{D}\mathbf{v}$ into the momentum balance equation yields the well known Navier–Stokes equation for incompressible fluid. (See e.g. [2, Sec. 3.3].) In practice, however, many fluids exhibit nonlinear dependence of tensor $\mathbb{S}$ on tensor $\mathbb{D}\mathbf{v}$. In the so called generalized Newtonian fluids, $\mathbb{S}$ is also related to $\mathbb{D}$ through the formula $\mathbb{S}(\mathbb{D}\mathbf{v}) = \beta\,\mathbb{D}\mathbf{v}$, but the coefficient β may depend on $|\mathbb{D}\mathbf{v}|$. (Since $|\mathbb{D}\mathbf{v}| = \sqrt{-2I_2(\mathbb{D}\mathbf{v})}$, where $I_2(\mathbb{D}\mathbf{v})$ is the second principal invariant of $\mathbb{D}\mathbf{v}$, β in fact depends on $\mathbb{D}\mathbf{v}$ through the invariant $I_2(\mathbb{D}\mathbf{v})$.)

In this paper, we deal with steady flows of incompressible generalized Newtonian fluids. Due to technical reasons, we write $f(|\mathbb{D}\mathbf{v}|^2)$ instead of $\beta(|\mathbb{D}\mathbf{v}|)$. Thus, the dynamic stress tensor $\mathbb{S}$ depends on the rate of deformation tensor $\mathbb{D}\mathbf{v}$ according to the law

$$\mathbb{S}(\mathbb{D}\mathbf{v}) := f\left(|\mathbb{D}\mathbf{v}|^2\right)\mathbb{D}\mathbf{v}. \tag{1.2}$$

1.2. The boundary-value problem

The steady state equation of balance of momentum has the form

$$-\operatorname{div}\mathbb{S}(\mathbb{D}\mathbf{v}) + \mathbf{v}\cdot\nabla\mathbf{v} + \nabla q \;=\; \mathbf{f}, \tag{1.3}$$

and the condition of incompressibility is

$$\operatorname{div}\mathbf{v} \;=\; 0. \tag{1.4}$$

Eqs.(1.3) and (1.4) are considered in a bounded Lipschitzian domain $\Omega \subset \mathbb{R}^N$, where $N = 2$ or $N = 3$. The unknowns are $\mathbf{v} \equiv (v_1, v_2, v_3)$ (the velocity) and q (the pressure). Function $\mathbf{f}$ on the right hand side of (1.3) represents the external body force. The density is, for simplicity, considered to be equal to 1. The system (1.3), (1.4) is completed by the homogeneous Dirichlet boundary condition (also called the “no slip condition”)

$$\mathbf{v} \;=\; \mathbf{0} \qquad \text{on } \partial\Omega. \tag{1.5}$$

In the rest of the paper, we use the abbreviation “BVP” for “boundary-value problem”. Foundations of the qualitative theory of the BVP (1.3)–(1.5) and related models were given in the papers [3] (by J. Nečas, J. Málek and M. Růžička) and [4] (by H. Bellout, F. Bloom and J. Nečas). The existence of a weak solution to the BVP (1.3)–(1.5) was proved by J. Frehse, J. Málek and M. Steinhauer [5] for general $N \geq 2$ under the condition that tensor $\mathbb{S}$ has the so called p-structure, where

$$\frac{2N}{N+2} \;<\; p \;<\; \infty. \tag{1.6}$$

(See Section 1.4 for the explanation what the p-structure means.) The proof is based on the method of Lipschitz truncations. The procedure is also explained in detail in paper [6]. (Here, the author extends the existential result to the case $p = 2N/(N+2)$, $N \geq 3$.)

1.3. Aims and results of this paper, previous related results

Since steady solutions to the Navier–Stokes equations are generally not unique, it is realistic to expect (although it has not yet been explicitly shown) that solutions to the BVP (1.3)–(1.5) are also not generally