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Wave breaking and global existence for a family of peakon equations with high order nonlinearity

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ABSTRACT

This paper is concerned with the Cauchy problem for a family of peakon equations with high order nonlinearity. To begin with, the local well-posedness results in Besov and Sobolev spaces are discussed. Then we obtain the precise blow-up scenario for strong solutions to the equation, which is conjectured by Anco et al., (2015). Moreover, we establish some global existence results for the strong solutions by deriving two useful conservation laws. Furthermore, wave breaking phenomenon is studied and we construct a new finite time blow-up solution to the equation with respect to the initial data.

1. Introduction

In this paper, we consider a family of peakon equations with high order nonlinearity as follows:

$$u_t - u_{txx} + (b+c)u^a u_x = bu^{a-1}u_x u_{xx} + cu^a u_{xxx}. \tag{1.1}$$

Its equivalent form reads:

$$m_t + cu^a m_x + bu^{a-1}u_x m = 0, \qquad m = u - u_{xx}, \tag{1.2}$$

where $t > 0$, $x \in \mathbb{R}$, and three parameters $a > 0$, $b \in \mathbb{R}$, and $c \neq 0$.

Eq. (1.1) was recently proposed in [1], and its single peakon and multi-peakon solutions are admitted if and only if $a \geq 0$ and $c \neq 0$. More precisely, the single peakon solutions to (1.2) have the following form:

$$u = u(x - vt) = \left(\frac{v}{c}\right)^{\frac{1}{a}} e^{-|x-vt|}, \tag{1.3}$$

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where $v = constant$ is the wave speed. The N-peakon solutions ($N \geq 2$) to Eq. (1.2) have been also obtained in [1]:

$$u(t,x) = \sum_{i=1}^{N} \alpha_i(t) e^{-|x-\beta_i(t)|}, \tag{1.4}$$

where the amplitudes $\alpha_i(t)$ and positions $\beta_i(t)$ satisfy the system of ODEs:

$$\begin{cases} \alpha_i' = (b-ac)\alpha_i \left(\alpha_j + \sum_{\substack{j=1 \\ j\neq i}}^{N} \alpha_j e^{-|\beta_i-\beta_j|} \right)^{a-1} \sum_{\substack{k=1 \\ k\neq i}}^{N} sgn(\beta_i-\beta_k)\alpha_k e^{-|\beta_i-\beta_k|}, \\ \beta_i' = c \left(\alpha_j + \sum_{\substack{j=1 \\ j\neq i}}^{N} \alpha_j e^{-|\beta_i-\beta_j|} \right)^{a}. \end{cases}$$

For $a = c = 1$, Eq. (1.2) reduces the b-family equation with quadratic nonlinearity, which can be derived as the family of asymptotically equivalent shallow water wave equations [2]. It admits peakon solutions for any constant b [3]. The local well-posedness, blow-up phenomena and global solutions for the b-family equation have been studied in [4]. By using Painlevé analysis [3,5], there are only two integrable equations in the b-family equation, that is, the Camassa–Holm equation (when $b = 2$) and the Degasperis–Procesi equation (when $b = 3$), which will be introduced as follows.

For $a = c = 1$ and $b = 2$, Eq. (1.2) becomes the Camassa–Holm (CH) equation which models the unidirectional propagation of shallow water waves over a flat bottom [6,7], and is also recognized as a model for the propagation of axially symmetric waves in hyperelastic rods [8]. It has a bi-Hamiltonian structure and is completely integrable [6]. Its solitary waves vanishing at both infinities are peakons [9] and they are orbitally stable [10]. It is also worth pointing out that the peakons replicate a feature that is characteristic for the waves of great height—waves of the largest amplitude that are exact traveling wave solutions of the governing equations for irrotational water waves, cf. [11]. The Cauchy problem and initial boundary value problem for the CH equation have been studied extensively [12–16]. It has both global strong solutions [12–14] and blow-up solutions at a finite time [12–15]. On the other hand, it also has global weak solutions [17,18]. In comparison with the celebrated KdV equation, the advantage of the CH equation lies in the fact that the CH equation has peakons [9] and models wave breaking [15] (namely, the wave remains bounded while its slope becomes unbounded in finite time [19]).

For $a = c = 1$ and $b = 3$, Eq. (1.2) becomes another integrable model for nonlinear shallow water dynamics, the so called Degasperis–Procesi (DP) equation [7,5]. It was proved in [3] that the DP equation has a bi-Hamiltonian structure and an infinite number of conservation laws, and admits peakon solutions which are analogous to the CH peakons. The Cauchy problem and initial boundary value problem for the DP equation have been studied extensively in [20–22,16,23–25]. Although the DP equation is very similar to the CH equation in the aspects of integrability, particularly in the form of equation, there are some significant differences between these two equations. One of the remarkable features of the DP equation is that it has not only (periodic) peakon solutions [3,25], but also (periodic) shock peakons [22,26]. Besides, the CH equation is a re-expression of geodesic flow on the diffeomorphism group [27], while the DP equation is regarded as a non-metric Euler equation [28].

For $a = 2$, $b = 3$ and $c = 1$, Eq. (1.2) reduces the Novikov equation with cubic nonlinearity. The Novikov equation is also integrable and was proposed in [29]. Its Lax pair, bi-Hamiltonian structure, and peakon solutions have been shown in [30,29]. The stability of peakons for the Novikov equation was proved in [31]. The local well-posedness, global and blow-up strong solutions, and global weak solutions for the Cauchy problem of the Novikov equation have been studied in [20,32].