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On the finite time singularities for a class of Degasperis–Procesi equations

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ABSTRACT

As we all know, the conservation laws take an important part in deriving wave breaking for Degasperis–Procesi (DP) type equation. In the article, we give a new method to study singularities in finite time for a class of DP equations with high nonlinear terms. The paper tells us that the result of wave breaking can be established without the conservation law.

1. Introduction

In this article, we consider wave breaking of the initial value problem for the following nonlinear partial differential equation

$$\begin{cases} \partial_t u + u\partial_x u + (1-\partial_x^2)^{-1}[F(u) + \partial_x G(u)] = 0, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

which can be regarded as a model for describing waves breaking. Here $u = u(t,x)$ stands for the fluid velocity at time t in the spatial x direction, F, G is a polynomial and the subscript denotes the partial derivative.

Eq. (1.1) can be considered as a type of Whitham equation [1,2]. The KdV equation cannot describe wave breaking, which is observed by many physicists in experiment. As Whitham [3] noted “it is intriguing to

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know what kind of simpler mathematical equation could include breaking and peaking", more details can be found in [3].

Let $F(u) = 0$, and $G(u) = ku$. Then Eq. (1.1) becomes the Fornberg–Whitham equation

$$\partial_t u + u\partial_x u + (1 - \partial_x^2)^{-1}\partial_x[ku] = 0, \quad t > 0,\ x \in \mathbb{R},$$

i.e.

$$\partial_t u - \partial_t u_{xx} + ku_x + uu_x = 3u_x u_{xx} + uu_{xxx}, \quad t > 0,\ x \in \mathbb{R},$$

which was introduced by Fornberg and Whitham in 1967 as a model for describing waves breaking (i.e., the solution remains bounded but its slope becomes infinite) [1,2].

If $F(u) = 0$ and $G(u) = \frac{3}{2}u^2$, Eq. (1.1) becomes the Degasperis–Procesi (DP) equation, which was derived by Degasperis and Procesi [4] in 1999 as a model for nonlinear shallow water dynamics [5]. It has a bi-Hamiltonian structure [6], a Lax pair based on a linear spectral problem of second order and is completely integrable [6]. One of the important features of the DP equation is that it has not only peakon solitons [6], which are given by

$$u(t,x) = ce^{-|x-ct|}$$

and periodic peakon solutions [7]

$$u(t,x) = \frac{\cosh(x - t - [x - t] - \frac{1}{2})}{\sinh \frac{1}{2}},$$

but also has shock peakons [8,9]

$$u(t,x) = -\frac{1}{t+k}\operatorname{sgn}(x)e^{-|x|}, \qquad k > 0,$$

and periodic shock peakons [10]

$$u_c(t,x) = \begin{cases} \left(\frac{\cosh \frac{1}{2}}{\sinh \frac{1}{2}} t + c\right)^{-1} \frac{\sinh(x - [x] - \frac{1}{2})}{\sinh \frac{1}{2}}, & x \in \mathbb{R}/\mathbb{Z},\ c > 0, \\ 0, & x \in \mathbb{Z}. \end{cases}$$

After the DP equation was derived, many results were obtained: Dullin, Gottwald and Holm [11] showed that the DP equation can be derived from the shallow water elevation equation by an appropriate Kodama transformation. Holm and Staley [5] studied the stability of solitons and peakons numerically. Lundmark and Szmigielski [12] also presented an inverse scattering approach for computing n-peakon solutions to the DP equation. Recently, Yin established local well-posedness of DP equation with initial datum $u_0 \in H^s, s > \frac{3}{2}$ on the line [13] and on the circle [14] by Kato's semigroup theory. The global existence of strong solutions and global weak solutions to DP equation were also investigated in [7,10,15–17]. It should be emphasized that by the conservation

$$H(t) =: \int yv dx = \int y_0 v_0 dx =: H_0 \tag{1.2}$$

with $y = (1 - \partial_x^2)u$ and $v = (4 - \partial_x^2)^{-1}u$, Liu and Yin etc. show that the DP equation not only has global existence of strong solution, but also has many kinds of blow-up phenomena, if the initial datum satisfies one of the following conditions:

(i). If the initial datum $u_0(x)$ is odd and $u'(0) < 0$ [13].