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Global dynamics of a quasilinear chemotaxis model arising from tumor invasion

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ABSTRACT

This paper is concerned with a quasilinear chemotaxis system

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v + wz, & x \in \Omega,\ t > 0, \\ w_t = -wz, & x \in \Omega,\ t > 0, \\ z_t = \Delta z - z + u, & x \in \Omega,\ t > 0, \end{cases}$$

with homogeneous Neumann boundary conditions in a smooth bounded domain $\Omega \subset \mathbb{R}^n (n \geq 1)$, where D satisfies $D(u) > 0$ for all $u \geq 0$ and behaves algebraically as $u \to \infty$. It is shown that if $\frac{S(u)}{D(u)} \leq Cu^\alpha$ with some constants $C > 0$ for all $u \geq 1$ and

$$\begin{cases} \alpha < 1 + \dfrac{1}{n}, & \text{if } 1 \leq n \leq 3, \\ \alpha < \dfrac{4}{n}, & \text{if } n \geq 4, \end{cases}$$

then for sufficiently smooth initial data, the system possesses a unique bounded classical solution (u, v, w, z), which exponentially converges to the equilibrium $(\bar{u}_0, \bar{v}_0 + \bar{w}_0, 0, \bar{u}_0)$ as $t \to +\infty$, where $\bar{u}_0 = \frac{1}{|\Omega|}\int_\Omega u_0(x)dx$, $\bar{v}_0 = \frac{1}{|\Omega|}\int_\Omega v_0(x)dx$ and$\bar{w}_0 = \frac{1}{|\Omega|}\int_\Omega w_0(x)dx$.

1. Introduction and main results

In this paper, we consider the initial–boundary value problem for the quasilinear chemotaxis system

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v + wz, & x \in \Omega,\ t > 0, \\ w_t = -wz, & x \in \Omega,\ t > 0, \\ z_t = \Delta z - z + u, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = \dfrac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ (u, v, w, z)(x, 0) = (u_0, v_0, w_0, z_0)(x), & x \in \Omega, \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, and $\frac{\partial}{\partial\nu}$ denotes the outward normal derivative on $\partial\Omega$. The system (1.1) was recently proposed by Fujie et al. [1] to describe a tumor invasion phenomenon with chemotaxis effect of Chaplain and Anderson type [2]. The unknown functions u, v, w and z denote the concentration of tumor cells, active extracellular matrix (ECM*), extracellular matrix (ECM) and matrix degrading enzymes (MDE), respectively. The main feature of system (1.1) is that the chemotactic cue is not released by the cells themselves, which has been called indirect chemotaxis model [3,4].

It has been proved in [5,6] that the indirect chemotaxis mechanism in system (1.1) has a role in enhancing the regularity and boundedness properties of solutions. Precisely, when $D(u) = 1$ and $S(u) = u$, using the semigroup estimate method, it has been shown in [5,6] that when $n \leq 3$ and the initial data satisfies

$$(u_0, v_0, w_0, z_0) \in C^0(\bar{\Omega}) \times W^{1,\infty}(\Omega) \times C^1(\bar{\Omega}) \times C^0(\bar{\Omega}) \text{ with } u_0, v_0, w_0, z_0 \geq 0, \tag{1.2}$$

there exists a unique nonnegative global classical solution (u, v, w, z) exponentially converging to the equilibrium $(\bar{u}_0, \bar{v}_0 + \bar{w}_0, 0, \bar{u}_0)$ as $t \to +\infty$, where $\bar{u}_0 = \frac{1}{|\Omega|}\int_\Omega u_0(x)dx$, $\bar{v}_0 = \frac{1}{|\Omega|}\int_\Omega v_0(x)dx$ and $\bar{w}_0 = \frac{1}{|\Omega|}\int_\Omega w_0(x)dx$. It is different from the direct chemotaxis model [7]

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla z), & x \in \Omega, t > 0, \\ z_t = \Delta z - z + u, & x \in \Omega, t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ (u, z)(x, 0) = (u_0, z_0)(x), & x \in \Omega, \end{cases} \tag{1.3}$$

whose solution will globally exist or blow up in a finite/infinite time depending strongly on the space dimension when $D(u) = 1$ and $S(u) = u$ (no blow-up in 1-D [8,9], critical mass blow-up in 2-D [10–15] and, generically blow-up in $\geq$ 3-D [16,17]).

As pointed out in [18], the chemotaxis model with nonlinear diffusion and cross-diffusion may play a predominant role in the solution behavior. For the direct chemotaxis model (1.3), the early literature has already contained some pieces of evidence confirming the intuitive idea that the tendency toward blow-up can be weakened at large cell densities if either cross-diffusion is inhibited, or diffusion is enhanced. More precisely, the boundedness or blow-up of solutions strongly depends on the space dimensions and the power α of the ratio $\frac{S(u)}{D(u)} \approx u^\alpha$ for large values of u. If $\alpha > \frac{2}{n}$, then there exist some solutions that will blow up in finite time or infinite time [19–23], whereas if $0 < \alpha < \frac{2}{n}$ and

$$\frac{S(u)}{D(u)} \leq C(u+1)^\alpha \quad \text{for all} \quad u \geq 0, \tag{1.4}$$

then the solution will globally exist with D behaves algebraically [24,25] or exponentially [26,27]. The results in [19–22,24–26] indicate that the power-type asymptotic behavior $\frac{S(u)}{D(u)} \approx u^{\frac{2}{n}}$ for the direct Keller–Segel chemotaxis model (1.3) is critical.

In contrast to the well-understood direct chemotaxis model (1.3), the theoretical understanding is much less developed in situations when a chemotactic cue is not released by the cells themselves as system (1.1). Specially, a complete understanding of the competitive interplay among diffusion, cross-diffusion and indirect chemotaxis is yet lacking. It is the purpose of the present work to achieve some insight into possible features of chemotaxis models accounting for indirect signal production mechanisms. More precisely, we will show how the indirect chemotaxis mechanisms affect the value of α in (1.4) to global classical solution. As in [24,25], we assume that the nonlinear diffusion $D(u)$ and chemosensitivity $S(u)$ satisfy

$$D, S \in C^2([0,\infty)), \ D(u) > 0 \ \text{ and } \ S(u) \geq 0 \ \text{ for all } \ u \geq 0, \tag{1.5}$$

and

$$K_0(u+1)^{m-1} \leq D(u) \leq K_1(u+1)^{M-1} \ \text{ for all } \ u \geq 0 \tag{1.6}$$

for some constants $m \in \mathbb{R}$, $M \in \mathbb{R}$, $K_0 > 0$ and $K_1 > 0$. Then we have the following main results.