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Existence results for second-order boundary-value problems with variable exponents

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ABSTRACT

Differential equations and variational problems with variable exponent arise from the nonlinear elasticity theory and the theory of electrorheological fluids. This paper presents several sufficient conditions for the existence of at least one weak solution for the following boundary value problem involving an ordinary differential equation with $p(x)$-Laplacian operator, and nonhomogeneous Neumann conditions

$$\begin{cases} -\left(|u'(x)|^{p(x)-2}u'(x)\right)' + \alpha(x)|u(x)|^{p(x)-2}u(x) = \lambda f(x, u(x)) & \text{in } (0,1), \\ |u'(0)|^{p(0)-2}u'(0) = -\lambda g(u(0)), \\ |u'(1)|^{p(1)-2}u'(1) = \lambda h(u(1)) \end{cases}$$

where $p \in C([0,1], \mathbb{R})$, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $g, h : \mathbb{R} \to \mathbb{R}$ are nonnegative continuous functions, $\lambda > 0$, $\alpha \in L^1([0,1])$, with $essinf_{[0,1]}\alpha > 0$. Our technical approach is based on variational methods. Some recent results are extended and improved. Moreover, a concrete example of an application is presented.

1. Introduction

In this paper, we consider the following boundary value problem involving an ordinary differential equation with $p(x)$-Laplacian operator, and nonhomogeneous Neumann conditions

$$\begin{cases} -\left(|u'(x)|^{p(x)-2}u'(x)\right)' + \alpha(x)|u(x)|^{p(x)-2}u(x) = \lambda f(x, u(x)) & \text{in } (0,1), \\ |u'(0)|^{p(0)-2}u'(0) = -\lambda g(u(0)), \\ |u'(1)|^{p(1)-2}u'(1) = \lambda h(u(1)) \end{cases} \qquad (P_\lambda^f)$$

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where $p \in C([0,1], \mathbb{R})$, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, (that is $x \to f(x,t)$ is measurable for all $t \in \mathbb{R}$, $t \to f(x,t)$ is continuous for almost every $x \in [0,1]$), $g, h : \mathbb{R} \to \mathbb{R}$ are nonnegative continuous functions, $\lambda > 0$ and $\alpha \in L^1([0,1])$, with $\alpha_- := ess\inf_{\in[0,1]} \alpha(x) > 0$.

The $p(x)$-Laplacian operator possesses more complicated nonlinearities than the p-Laplacian operator, mainly due to the fact that it is not homogeneous. In recent years, the investigation of differential equations and variational problems with variable exponent has become a new and interesting topic. It arises from the nonlinear elasticity theory, the theory of electrorheological fluids, etc (see [1]). The first important discovery on electrorheological fluids was contributed by Willis Winslow in 1949. The viscosity of these fluids depends upon the electric field of the fluids. He discovered that the viscosity of such fluids as instance lithium polymetachrylate in an electrical field is an inverse relation to the strength of the field. The field causes string-like formations in the fluid, parallel to the field. They can increase the viscosity five orders of magnitude. This event is called the Winslow effect. For a general account of the underlying physics see [2] and for some technical applications [3]. Electrorheological fluids also have functions in robotics and space technology. Many experimental researches have been done chiefly in the USA, as in NASA laboratories. Problems with variable exponent also have extensive applications in various research fields, such as the image-processing model (see, e.g., [4,5]), stationary thermorheological viscous flows (see [6]), and the mathematical description of the processes of filtration of ideal barotropic gases through porous media (see [7]).

The study of various mathematical problems with variable exponent has received considerable attention in recent years. For background and recent results, we refer the reader to [8–22] and the references therein for details. For example, Zhang in [22] via Leray–Schauder degree, obtained sufficient conditions for the existence of one solution for a weighted $p(x)$-Laplacian system boundary value problem. Yao in [21] by using the variational method, under appropriate assumptions on f and g, obtained a number of results on existence and multiplicity of solutions for the nonlinear Neumann boundary value problem of the following form

$$\begin{cases} -div(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u = \lambda f(x,u), & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\dfrac{\partial u}{\partial v} = \mu g(x,u), & \text{on } \partial\Omega \end{cases}$$

where $\lambda, \mu \in \mathbb{R}$, $p \in C(\bar{\Omega})$ and $p(x) > 1$. Cammaroto et al. in [12] by using a three critical points theorem due to Ricceri, obtained the existence of three weak solutions for a Neumann problem involving the $p(x)$-Laplacian. D'Aguì in [13] by using variational methods, established the existence of an unbounded sequence of weak solutions for following problem

$$\begin{cases} -\left(|u'(x)|^{p(x)-2}u'(x)\right)' + \alpha(x)|u(x)|^{p(x)-2}u(x) = \lambda f(x,u(x)) & \text{in } (0,1), \\ |u'(0)|^{p(0)-2}u'(0) = -\mu g(u(0)), \\ |u'(1)|^{p(1)-2}u'(1) = \mu h(u(1)) \end{cases}$$

where all the data are as in the problem (P_λ^f) and $\mu \geq 0$. In [16] based on the variational methods and critical-point theory the existence of at least three solutions for elliptic problems driven by a $p(x)$-Laplacian was established. The existence of at least one nontrivial solution was also proved. In [11] Bonanno et al., studied a class of nonlinear differential boundary value problems with variable exponent, and established the existence of at least one non-zero solution, without assuming on the nonlinear term any condition either at zero or at infinity. The approach was developed within the framework of the Orlicz–Sobolev spaces with variable exponent and it was based on a local minimum theorem for differentiable functions.

Motivated by the above facts, in the present paper, we study the existence of at least one non-trivial weak solution for the problem (P_λ^f) under an asymptotical behaviour of the nonlinear datum at zero, see Theorem 3.1. Example 3.2 illustrates Theorem 3.1. We give some remarks on our results. In Theorem 3.7