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Optimal convergence rates of the supercritical surface quasi-geostrophic equation

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ABSTRACT

This study is concerned with the optimal convergence rates of the supercritical surface quasi-geostrophic equation. When the global weak solution θ of the super critical surface quasi-geostrophic equation satisfies

$$\nabla\theta \in L^r(0,\infty;\dot{B}^0_{p,\infty}(\mathbb{R}^2)) \quad \text{for } \frac{2}{p}+\frac{\alpha}{r}=\alpha, \quad \frac{2}{\alpha}<p<\infty,$$

then even for large initial data perturbation, every weak solution $\tilde{\theta}(x,t)$ of the perturbed quasi-geostrophic equation converges to $\theta(x,t)$ with the optimal algebraic convergence rate

$$\|\tilde{\theta}(t)-\theta(t)\|_{L^2} \le C(1+t)^{-\frac{1}{\alpha}} \qquad t>0.$$

The findings are mainly based on some new estimates for the trilinear form in Besov spaces and the generalized Fourier splitting method.

1. Introduction

In this study, we consider the Cauchy problem of the supercritical dissipative surface quasi-geostrophic fluid motion model given by Constantin, Majda and Tabak [1]

$$\begin{cases} \left(\dfrac{\partial}{\partial t}+u\cdot\nabla\right)\theta+\kappa\Lambda^\alpha\theta=f, & (x,t)\in\mathbb{R}^2\times(0,\infty), \\ \theta(x,0)=\theta_0, & x\in\mathbb{R}^2. \end{cases} \tag{1.1}$$

Here $0<\alpha<1$ and $\kappa>0$ is a dissipative coefficient. Λ is the Riesz potential operator defined by $\Lambda=(-\Delta)^{1/2}$, $\theta(x,t)$ is an unknown scalar function representing potential temperature. $u(x,t)$ is the velocity field determined by

$$u=-\nabla^\perp(-\Delta)^{-\frac{1}{2}}\theta=-\mathcal{R}^\perp\theta=(\mathcal{R}_2\theta,-\mathcal{R}_1\theta) \tag{1.2}$$

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where $\mathcal{R}_j$, $j=1,2$, is the 2D Riesz transform.

Due to its mathematical importance and its potential for applications in atmospheric and oceanographic sciences [2], on one hand, there is a large literature on the well-posedness for solutions of the surface quasi-geostrophic equation. When $\alpha=1$, the surface quasi-geostrophic equation (1.1) shares similar features with the three-dimensional incompressible Navier–Stokes equations. Thus $\alpha=1$ is therefore referred as the critical case, while the cases $0<\alpha<1$ and $1<\alpha\leq 2$ are supercritical and subcritical, respectively. In the critical and subcritical cases, the global smooth solution for the sufficiently smooth initial data have been proved by Constantin and Wu [3], Caffarelli and Vasseur [4], Kiselev, Nazarov and Volberg [5], Constantin and Vicol [6], respectively. However, the question of global existence for smooth solution of the supercritical quasi-geostrophic equation is still unanswered. One may refer to many important studies on global solutions in some critical spaces or regularity criteria of weak solutions have been examined (see [7–12] and references therein).

On the other hand, it is desirable to understand the asymptotic behavior of the surface quasi-geostrophic equation (1.1). By using the classic Fourier splitting methods, Constantin and Wu [3] firstly proved the L^2 decay of weak solution of zero-forced quasi-geostrophic equation (1.1)

$$\|\theta(t)\|_{L^2}\leq C(1+t)^{-\frac{1}{\alpha}}. \tag{1.3}$$

The above result essentially shows there exists a global weak solution of (1.1) that converges to the trivial solution $\bar{\theta}\equiv 0$ with the rate $(1+t)^{-\frac{1}{\alpha}}$. The result is further studied by many authors such as Dong and Liu [13], Carrillo [14], Ferreira, Niche and Planas [15], Niche and Schonbek [16], Zhou [17] and references therein.

When $f\neq 0$ in (1.1), it is clear that (1.1) has no any constant solutions. It is natural and important to consider the asymptotic convergence issue of nontrivial solutions of the nonzero forced surface quasi-geostrophic equation (1.1). More precisely, we consider the following perturbed surface quasi-geostrophic equation

$$\begin{cases}\left(\dfrac{\partial}{\partial t}+v\cdot\nabla\right)\tilde{\theta}+\kappa\Lambda^{\alpha}\tilde{\theta}=f,\\ v=-\nabla^{\perp}(-\Delta)^{-\frac{1}{2}}\tilde{\theta}=-\mathcal{R}^{\perp}\tilde{\theta}=(\mathcal{R}_2\tilde{\theta},-\mathcal{R}_1\tilde{\theta}),\\ \tilde{\theta}(x,0)=\theta_0+w_0\end{cases} \tag{1.4}$$

where $w_0(x)$ is the initial data perturbation. On one hand, as for the uniform convergence, Chae and Lee [18] proved the global solution θ of the critical surface quasi-geostrophic equation converges to $\tilde{\theta}$ in Besov space $C([0,\infty);B^1_{2,1})\cap L^1((0,\infty);\dot{B}^2_{2,1})$ under the small initial perturbation. Dong and Chen [19] proved the global solution θ of the critical and supercritical surface quasi-geostrophic equation asymptotically converges to $\tilde{\theta}$ in critical Lebesgue space

$$\nabla\theta\in L^r(0,\infty;L^p(\mathbb{R}^2)),\quad \frac{2}{p}+\frac{\alpha}{r}=\alpha,\quad \text{and}\quad \frac{2}{\alpha}<p<\infty$$

for arbitrary initial data and external force perturbations. Ren and Ma [20] studied the global solution θ of the subcritical surface quasi-geostrophic equation asymptotically converges to $\tilde{\theta}$ in Lorentz space. Jia, Gui and Dong [21] further examined the asymptotic stability of the supercritical surface quasi-geostrophic equation in critical BMO spaces. The first and the third authors [22] recently examined the asymptotic convergence rates of the supercritical surface quasi-geostrophic equation in the critical Morrey space which is larger than Lorentz space and Lebesgue space. We would like to mention that Dai [23] proved the global solution θ of the subcritical surface quasi-geostrophic equation asymptotically converges to the solution of the following steady quasi-geostrophic equation

$$\begin{cases}(v\cdot\nabla)\,\tilde{\theta}+\kappa\Lambda^{\alpha}\tilde{\theta}=f,\\ v=-\nabla^{\perp}(-\Delta)^{-\frac{1}{2}}\tilde{\theta}=-\mathcal{R}^{\perp}\tilde{\theta}=(\mathcal{R}_2\tilde{\theta},-\mathcal{R}_1\tilde{\theta})\end{cases} \tag{1.5}$$