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Existence of positive solutions for a quasilinear Schrödinger equation

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ABSTRACT

This paper is concerned with the quasilinear Schrödinger equation

$$-\Delta u+V(x)u-\Delta\left[(1+u^2)^{\frac{1}{2}}\right]\frac{u}{2(1+u^2)^{\frac{1}{2}}}=\mu h(u) \quad \text{in } \mathbb{R}^N, \tag{0.1}$$

where $N\geq 3$ and $\mu>0$ is a parameter. Under some appropriate assumptions on the potential V and the nonlinear term h, we establish the existence of a positive solution for (0.1) with sufficiently large μ. Our method is based on a change of variables, monotonicity trick developed by Jeanjean and a priori estimate.

1. Introduction and main result

In recent years, the time-dependent quasilinear Schrödinger equation

$$i\partial_t z=-\Delta z+W(x)z-\tilde{h}(|z|^2)z-\Delta\left[\ell(|z|^2)\right]\ell'(|z|^2)z \tag{1.1}$$

has received considerable attention due to the fact that it can be seen as a model of various phenomena in physics, where $z:\mathbb{R}\times\mathbb{R}^N\to\mathbb{C}$, $W:\mathbb{R}^N\to\mathbb{R}$ is a given potential and $\tilde{h}$, $\ell:\mathbb{R}\to\mathbb{R}$ are suitable functions. In particular, (1.1) with $\ell(s)=s$ models the time evolution of the condensate wave function in super-fluid film, while (1.1) with $\ell(s)=(1+s)^{1/2}$ models the self-channeling of a high-power ultra short laser in matter. We refer the reader to [1–5] and references therein for more information on physical background of (1.1).

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It is known that, via the ansatz $z(t,x) = e^{-i\omega t}u(x)$, (1.1) can be reduced to the elliptic equation

$$-\Delta u + V(x)u - \Delta\big[\ell(u^2)\big]\ell'(u^2)u = h(u) \text{ in } \mathbb{R}^N, \tag{1.2}$$

where $V(x) = W(x) - \omega$ and $h(u) = \tilde{h}(u^2)u$. During the last twenty years, (1.2) with $\ell(s) = s$ has been studied extensively. Many existence and multiplicity results of nontrivial solutions have been established by differential methods such as minimizations, change of variables, Nehari method and perturbation method. See, for example, [6–15] and references therein. In the current paper, we are interested in another special case, that is $\ell(s) = (1+s)^{1/2}$, in which less results are known and we are only aware of the papers [16–18] in this direction. These papers deal with problem (1.2) with $\ell(s) = (1+s)^{1/2}$ and a special nonlinearity or a general nonlinearity satisfying global assumptions including the well known Ambrosetti–Rabinowitz condition. A natural question is that whether or not we can obtain existence results of nontrivial solutions for (1.2) without any global assumption on the nonlinearity? In this paper, we continue the works of [16–18] and give an affirmative answer to the above interesting question. We will establish the existence of a positive solution for (1.2) with $\ell(s) = (1+s)^{1/2}$ and a nonlinear term only satisfying suitable conditions in a neighborhood of zero. More precisely, we will deal with the quasilinear Schrödinger equation

$$-\Delta u + V(x)u - \Delta\big[(1+u^2)^{\frac{1}{2}}\big]\frac{u}{2(1+u^2)^{\frac{1}{2}}} = \mu h(u) \text{ in } \mathbb{R}^N, \tag{1.3}$$

where $N \geq 3$, $V \in C^1(\mathbb{R}^N, \mathbb{R})$, $h \in C(\mathbb{R}, \mathbb{R})$ and $\mu > 0$ is a parameter. We make the following assumptions on the potential V and the nonlinear term h:

(V_1) $V(x) = V(|x|)$ and $0 < \alpha \leq V(x) \leq \beta < \infty$ for $x \in \mathbb{R}^N$;

(V_2) there exists $A \in \big[0, \frac{(N-2)^2}{2}\big)$ such that $|\nabla V(x) \cdot x| \leq \frac{A}{|x|^2}$ for $x \in \mathbb{R}^N \setminus \{0\}$;

(h_1) $h(t) = 0$ for $t \leq 0$ and there exists $q \in \left(2, \frac{2N}{N-2}\right)$ such that

$$-\infty < \liminf_{t\to 0^+} \frac{h(t)}{t^{q-1}} \leq \limsup_{t\to 0^+} \frac{h(t)}{t^{q-1}} < +\infty;$$

(h_2) there exists $p \in \left(2, \frac{2N}{N-2}\right)$ such that $\liminf_{t\to 0^+} \frac{H(t)}{t^p} > 0$, where $H(t) = \int_0^t h(s)\,ds$.

From (h_1) and (h_2), it can be seen that $q \leq p$. We also want to point out that, unlike the aforementioned papers, no condition is assumed on the nonlinear term h near infinity.

The main result of this paper reads as follows.

Theorem 1.1. *Suppose that $(V_1) - (V_2)$ and $(h_1) - (h_2)$ hold. Then there exists $\bar{\mu} > 0$ such that, for any $\mu > \bar{\mu}$, problem (1.3) has at least a positive solution.*

Remark 1.2. It is clear that positive constant potentials satisfy (V_1) and (V_2). Another typical example satisfying (V_1) and (V_2) is

$$V(x) = A_1 + \frac{A_2}{1 + A_3|x|^2},$$

where $A_1 > 0$, $A_3 > 0$ and $-\min\big\{A_1, \frac{A_3(N-2)^2}{4}\big\} < A_2 < \frac{A_3(N-2)^2}{4}$ are constants.

Remark 1.3. In [19,20], the authors use (h_1) and (h_2) together with a local Ambrosetti–Rabinowitz condition to prove the existence and multiplicity of nontrivial solutions for semilinear elliptic equations. It is a little surprising that neither global nor local Ambrosetti–Rabinowitz condition is assumed in Theorem 1.1.