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On a singular limit for stratified compressible fluids

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ABSTRACT

We consider a singular limit problem for the complete compressible Euler system in the low Mach and strong stratification regime. We identify the limit problem – the anelastic Euler system – in the case of well prepared initial data. The result holds in the large class of the dissipative measure-valued solutions of the primitive system. Applications are discussed to the driven shallow water equations.

1. Introduction

The following system of equations arises in a number of real world applications, in particular in certain astrophysical and meteorological models (see e.g. the survey by Klein [1]):

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2}\nabla_x p(\varrho, \vartheta) = \frac{1}{\varepsilon^2}\varrho \nabla_x F, \tag{1.2}$$

$$\partial_t \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{1}{\varepsilon^2}\varrho e(\varrho,\vartheta) \right) + \operatorname{div}_x \left[\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{1}{\varepsilon^2}\varrho e(\varrho,\vartheta) \right) \mathbf{u} \right] + \frac{1}{\varepsilon^2}\operatorname{div}_x(p(\varrho,\vartheta)\mathbf{u}) = \frac{1}{\varepsilon^2}\varrho \nabla_x F \cdot \mathbf{u}. \tag{1.3}$$

The equations (1.1), (1.2), and (1.3) represent a mathematical formulation of the conservation of mass, momentum, and energy, respectively, of a compressible inviscid fluid driven by a potential force $\nabla_x F$. Here, the state of the fluid at a time t and a spatial position x is given by its mass density $\varrho = \varrho(t,x)$, the macroscopic velocity $\mathbf{u} = \mathbf{u}(t,x)$ and the (absolute) temperature $\vartheta = \vartheta(t,x)$. The pressure $p = p(\varrho,\vartheta)$ and the internal energy density $e = e(\varrho,\vartheta)$ are given explicitly through an equation of state.

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To close the system we specify the physical domain — an infinite slab, "periodic" in the horizontal variable:

$$\Omega = \left([-1,1]|_{\{-1,1\}} \right)^2 \times [0,1],$$

supplemented with the impermeability condition

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \text{ meaning, } u^3(t,x_1,x_2,0) = u^3(t,x_1,x_2,1) = 0. \tag{1.4}$$

Later, to extend the range of possible applications of our result, we consider a slightly more general setting, $\Omega \subset R^N$, $N = 2,3$, – a bounded regular domain – supplemented with the impermeability condition

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{1.5}$$

Problem (1.2)–(1.4) contains a small positive parameter $\varepsilon > 0$. Our aim is to identify the limit problem for $\varepsilon \to 0$. Rather surprisingly, the limit problem is not unique and depends on the choice of the initial data. To see this, let us examine the associated static system

$$\nabla_x p(\varrho, \vartheta) = \varrho \nabla_x F. \tag{1.6}$$

To simplify presentation, we suppose that p satisfies the standard Boyle–Mariotte law,

$$p(\varrho, \vartheta) = \varrho\vartheta,$$

and that

$$F(x) = -x_3.$$

Accordingly, the pressure p in (1.6) depends only on the vertical variable x_3 and problem (1.6) reduces to

$$\frac{1}{\varrho} \partial_{x_3}(\varrho\vartheta) = -1. \tag{1.7}$$

1.1. Isothermal limit

Suppose, that $\vartheta = \overline{\vartheta} > 0$ — a positive constant. Then the stationary problem (1.7) can be explicitly solved,

$$\varrho = \tilde{\varrho}(x_3), \quad \tilde{\varrho}(x_3) = c_M \exp\left(-\frac{x_3}{\overline{\vartheta}} \right), \ c_M > 0,$$

where the value of the constant c_M is uniquely determined by prescribing the total mass $M = \int_\Omega \tilde{\varrho} \ dx$. The limit system for the isothermal case has been identified in [2]. It turns out that the limit velocity field $\mathbf{U}$ has only two components,

$$\mathbf{U}(x_1,x_2,x_3) = \left[U^1(x_1,x_2,x_3), U^2(x_1,x_2,x_3), 0 \right] \equiv \left[\mathbf{U}_h(x_1,x_2,x_3), 0 \right],$$

where, for any fixed x_3, the field $\mathbf{U}_h(\cdot, x_3)$ satisfies the incompressible Euler system

$$\mathrm{div}_h \mathbf{U}_h = 0, \tag{1.8}$$

$$\partial_t \mathbf{U}_h + \mathbf{U}_h \cdot \nabla_h \mathbf{U}_h + \nabla_h \Pi = 0, \tag{1.9}$$