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Global strong solution for the full MHD equations with vacuum and large data

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A R T I C L E I N F O

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A B S T R A C T

This paper deals with the compressible, viscous and heat-conducting magnetohydrodynamic (MHD) equations in dimension one. In the case when all transport coefficients are positive constant, we establish the global existence and uniqueness of strong solution for the initial–boundary-value problem. The initial data is large and vacuum state is allowed.

1. Introduction

Magnetohydrodynamics (MHD) concerns the motion of conducting fluids in an electromagnetic field. It has a very broad range of applications, from liquid metals to cosmic plasmas; see, e.g., [1–3]. The following equations describe the compressible planar magnetohydrodynamic flows which implies that the flows are uniform in the transverse directions (cf. [4,5]):

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + (\rho u^2 + \rho\theta + \frac{1}{2}|\mathbf{b}|^2)_x = u_{xx}, \\ (\rho \mathbf{w})_t + (\rho u \mathbf{w} - \mathbf{b})_x = \mathbf{w}_{xx}, \\ \mathbf{b}_t + (u\mathbf{b} - \mathbf{w})_x = \mathbf{b}_{xx}, \\ (\rho\theta)_t + (\rho u\theta)_x + \rho\theta u_x = \theta_{xx} + u_x^2 + \mathbf{w}_x^2 + \mathbf{b}_x^2, \end{cases} \tag{1.1}$$

where ρ is the density, $u \in \mathbb{R}$ and $\mathbf{w} = (u^2, u^3) \in \mathbb{R}^2$ are the longitudinal velocity and the transverse velocity respectively, $\mathbf{b} = (b^2, b^3) \in \mathbb{R}^2$ is the transverse magnetic field, and θ is the absolute temperature. Here, all the transport coefficients and various gas constants are assumed to be 1, since they do not create any

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mathematical difficulty. In particular, we focus on the ideal gas and assume the pressure

$$P = P(\rho, \theta) = \rho\theta.$$

Consider Eqs. (1.1) in the region $\{(x,t) : x \in I = [0,1], t > 0\}$, with the initial functions

$$(\rho, u, \mathbf{w}, \mathbf{b}, \theta)|_{t=0} = (\rho_0, u_0, \mathbf{w}_0, \mathbf{b}_0, \theta_0)(x) \quad \text{for} \quad x \in I \tag{1.2}$$

and the boundary conditions

$$(u,\ \mathbf{w},\ \mathbf{b},\ \partial_x\theta)|_{x\in\partial I} = 0, \quad t \geq 0. \tag{1.3}$$

There have been a lot of studies on MHD by physicists and mathematicians because of its physical importance, complexity, rich phenomena, and mathematical challenges [1,2,4–11]. Below we give a few words on some existence results of solutions in dimension one. If no initial vacuum appears, the local in time existence is known from the Banach theorem and the contractibility of the operator defined by linearization of the problem (see [12]), and the global existence is due to Kazhikhov [13]. Besides, if the diffuse coefficients, especially the heat-conducting κ, depends on the density and the temperature, there are many other nice results for (1.1) concerning the existence, uniqueness and the Lipschitz continuous dependence, see, for existence, the papers [5–8,10] and cited therein.

The problem becomes more complicated when the initial vacuum is permitted. In the spirit of the paper [14] by Cho–Kim, the local existence of strong solutions for (1.1) was obtained by Fan–Yu [15] under the initial boundary layer conditions (see (1.6)). Assume that the heat-conducting coefficient satisfies

$$C^{-1}(1+\theta^q) \leq \kappa(\theta) \leq C(1+\theta^q), \quad q > 0. \tag{1.4}$$

Fan–Huang–Li [4] obtained the global existence of strong solution to (1.1)–(1.3). We mention that condition (1.4) plays an essential role in their proof. For example, in the inequality

$$\int_0^t \int_I \frac{u_x^2 + |\mathbf{w}_x|^2 + |\mathbf{b}_x|^2}{\theta^{\tilde{\alpha}}} + \int_0^t \int_I \frac{\kappa(\theta)\theta_x^2}{\theta^{1+\tilde{\alpha}}} \leq C + C\int_0^t \|\theta\|_{L^\infty(I)}^{1-\tilde{\alpha}}, \quad \tilde{\alpha} \in (0, \min(1,q)),$$

it requires $q \geq 1 - \tilde{\alpha}$ so that $\int_0^t \|\theta\|_{L^\infty(I)}^{1-\tilde{\alpha}}$ can be dominated by the left-hand side terms. In other words, the argument in [4] does not work any more when $q = 0$ in (1.4).

The goal of this paper is to obtain the global existence of strong solution to (1.1)–(1.3) in the case when all transport coefficients are constant. Our main result reads as follows.

Theorem 1.1. *Suppose (1.2) is compatible with (1.3), and the initial functions satisfy*

$$\begin{aligned} &0 < \int_I \rho_0 dx, \quad 0 \leq \rho_0 \in H^2(I), \\ &u_0 \in H^2(I), \quad \mathbf{w}_0,\ \mathbf{b}_0 \in H^2(I), \quad 0 \leq \theta_0 \in H^2(I). \end{aligned} \tag{1.5}$$

Suppose in addition that the following equalities hold true

$$\begin{aligned} &u_{0xx} - (\rho_0\theta_0 + \frac{1}{2}|\mathbf{b}_0|^2)_x = \sqrt{\rho_0}g_1, \\ &\mathbf{w}_{0xx} - \mathbf{b}_{0x} = \sqrt{\rho_0}g_2, \\ &\theta_{0xx} + u_{0x}^2 + |\mathbf{w}_{0x}|^2 + |\mathbf{b}_{0x}|^2 = \sqrt{\rho_0}g_3, \end{aligned} \tag{1.6}$$

with $g_1,\ g_2, g_3 \in L^2(I)$.