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Internal rogue waves in stratified flows and the dynamics of wave packets

T.Y. Liu[a], H.N. Chan[a], R.H.J. Grimshaw[b], K.W. Chow[a,*]

[a] Department of Mechanical Engineering, University of Hong Kong, Pokfulam, Hong Kong
[b] Department of Mathematics, University College London, Gower Street, London, WC1E 6BT, United Kingdom

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ABSTRACT

A theoretical study on the occurrence of internal rogue waves in density stratified flows is conducted. While internal rogue waves for long wave models have been studied in the literature, the focus here is on unexpectedly large amplitude displacements arising from the propagation of slowly varying wave packets. In the first stage of the analysis we calculate new exact solutions of the linear modal equations in a finite domain for realistic stratification profiles. These exact solutions are then used to facilitate the calculations of the second harmonic and the induced mean motion, leading to a nonlinear Schrödinger equation for the evolution of a wave packet. The dispersion and nonlinear coefficients then determine the likelihood for the occurrence of rogue waves. Several cases of buoyancy frequency (N) are investigated. For N^2 profiles of hyperbolic secant form, rogue waves are unlikely to occur as the dispersion and nonlinear coefficients are of opposite signs. For N^2 taking constant values, rogue waves will arise for reasonably small carrier envelope wavenumbers, in sharp contrast with the situation for a free surface, where the condition is $kh > 1.363$ (k = wavenumber of the carrier envelope, h = depth). Finally, a special N^2 profile permits an analytical treatment for a linear shear current. Unexpectedly large amplitude waves are possible as the dispersion and nonlinear coefficients can then be of the same sign.

1. Introduction

Stratified flows and internal waves occur frequently in the atmosphere and the oceans [1,2]. The dynamics and properties of such flows play an essential role in processes such as transport, mixing and the movement of nutrients in the oceans. Studies of small amplitude disturbances in stratified flows will then enhance the analytical description of the fluid motion, and have developed into a branch of classical hydrodynamic theory [3]. Concerning the dynamics of the oceans, a commonly used assumption is to employ the Boussinesq approximation, where the variation in the density is ignored except in the buoyancy term.

* Corresponding author.
E-mail address: kwchow@hku.hk (K.W. Chow).

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In this approximation the governing equation for small disturbances is defined as an eigenvalue problem for the phase speed:

$$(U-c)\left(\phi_{yy}-k^2\phi\right)-U_{yy}\phi+\frac{N^2\phi}{U-c}=0, \tag{1}$$

where ϕ, $U=U(y)$, k, c are the modal function of the linearized vertical velocity field, background shear, wavenumber and wave speed respectively, and y is the vertical coordinate. N is the buoyancy frequency given by

$$N^2=-\frac{g}{\overline{\rho}}\frac{d\,\overline{\rho}}{dy} \qquad (g \text{ is gravity}), \tag{2}$$

where $\overline{\rho}$ is the mean density profile in the undisturbed state. Extensive studies have been performed on this classical equation, ranging from special exact solutions [4] to stability considerations [5].

For the special case where a background shear flow is absent ($U(y)=0$), this eigenvalue problem becomes the modal equation

$$\phi_{yy}+\left[\frac{N^2}{c^2}-k^2\right]\phi=0, \tag{3}$$

which, together with the boundary conditions, defines an eigenvalue problem for the speed c with a given input wavenumber k. For stable stratification, there is no instability in the absence of a current. In developing the theory for the propagation of waves with a modal function given by Eq. (3), it is necessary to determine the dispersion relation $\omega=\omega(k)$ where ω is the wave frequency, the group velocity

$$c_g=\partial\omega/\partial k,$$

and frequently $c_{gk}=\partial^2\omega/\partial k^2$ as well. For this purpose, it is useful if explicit solutions of Eq. (3) can be found. One objective of this work is to establish exact solutions for this reduced form of the eigenvalue problem Eq. (3), in particular for a special class of density profiles, namely, the buoyancy frequency being the square of the hyperbolic secant with respect to the vertical coordinate. The underlying methodology is to note a connection with the nonlinear Schrödinger equation (NLSE) from the theory of solitons. Special solutions from the NLSE theory are then employed in solving the eigenvalue problem Eq. (3). While knowledge from classical linear differential equations, e.g. Pöschl–Teller and reflectionless potentials [6], hypergeometric and Legendre functions, could be invoked for this 'sech square' profile, utilizing coupled Schrödinger models can generate eventually solutions for more complicated, and even asymmetric, density profiles.

To describe waves with larger amplitude, a Hamiltonian formulation or higher order perturbation scheme will be necessary. A Hamiltonian approach of a two-layer fluid with nonzero mean flow can demonstrate the properties of wave–current interactions vividly [7]. Investigations of higher order series expansion, e.g. the Witting series and the Karabut system, can also be utilized to elucidate solitary waves for fluids of a finite depth [8]. Indeed ingenious mathematical methodologies have been applied to reveal intriguing nonlinear dynamics of these hydrodynamic systems, e.g. an implicit function approach is employed for the propagation of capillary–gravity waves in a spherical coordinate system [9].

Similar to the case of surface waves, the propagation of weakly nonlinear internal wave trains in a continuously stratified fluid is described by the NLSE [10]. Among various solutions of the NLSE which are physically relevant to water waves, the Peregrine breather (PB) solution [11] has attracted substantial attention recently due to its application to model rogue waves in the ocean [12]. The localized nature of the PB in both space and time resembles the character of a rogue wave as an entity which 'appears from nowhere and disappears without a trace'. Remarkably, PB and its higher order variations are realizable in water wave tanks [13,14]. In the context of surface waves, PB exists only in the focusing or deep water regime,