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# Lifespan of classical discontinuous solutions to the generalized nonlinear initial–boundary Riemann problem for hyperbolic conservation laws with small BV data: Rarefaction waves☆

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ABSTRACT

In the present paper the author investigates the generalized nonlinear initial–boundary Riemann problem with small BV data for general $n \times n$ quasilinear hyperbolic systems of conservation laws with nonlinear boundary conditions in a half space $\{(t,x)|t \geq 0, x \geq 0\}$, where the Riemann solution contains rarefaction waves. Combining the techniques proposed by Li and Kong with the modified Glimm's functional, the author obtains the almost global existence and lifespan of classical discontinuous solutions to a class of generalized nonlinear initial–boundary Riemann problem, which can be regarded as a small BV perturbation of the corresponding nonlinear initial–boundary Riemann problem. This result is also applied to the problem of planar steady supersonic Euler flow past a wedge.


## 1. Introduction and main result

Consider the following quasilinear hyperbolic system of conservation laws:

$$\partial_t u + \partial_x f(u) = 0, \qquad x \in \mathbf{R}, \ \ \mathbf{t > 0}, \tag{1.1}$$

where $u = (u_1, \ldots, u_n)^T$ is the unknown vector-valued function of $(t,x)$, $f : \mathbf{R^n} \to \mathbf{R^n}$ is a given $C^3$ vector function of $u$.

It is assumed that system (1.1) is strictly hyperbolic, i.e., for any given $u$ on the domain under consideration, the Jacobian $A(u) = \nabla f(u)$ has $n$ real distinct eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u). \tag{1.2}$$

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Let $l_i(u) = (l_{i1}(u), \dots, l_{in}(u))$ (resp. $r_i(u) = (r_{i1}(u), \dots, r_{in}(u))^T$) be a left (resp. right) eigenvector corresponding to $\lambda_i(u)$ $(i = 1, \dots, n)$:

$$l_i(u)A(u) = \lambda_i(u)l_i(u) \qquad (\text{resp. } A(u)r_i(u) = \lambda_i(u)r_i(u)). \tag{1.3}$$

We have

$$\det|l_{ij}(u)| \neq 0 \qquad (\text{equivalently, } \det|r_{ij}(u)| \neq 0). \tag{1.4}$$

Without loss of generality, we may assume that on the domain under consideration

$$l_i(u)r_j(u) \equiv \delta_{ij} \quad (i, j = 1, \dots, n) \tag{1.5}$$

and

$$r_i^T(u)r_i(u) \equiv 1 \quad (i = 1, \dots, n), \tag{1.6}$$

where $\delta_{ij}$ stands for the Kronecker's symbol.

Clearly, all $\lambda_i(u)$, $l_{ij}(u)$ and $r_{ij}(u)$ $(i, j = 1, \dots, n)$ have the same regularity as $A(u)$, i.e., $C^2$ regularity.

We also assume that on the domain under consideration, each characteristic field is either genuinely nonlinear in the sense of Lax (cf. [1]):

$$\nabla\lambda_i(u)r_i(u) \neq 0 \tag{1.7}$$

or linearly degenerate in the sense of Lax:

$$\nabla\lambda_i(u)r_i(u) \equiv 0. \tag{1.8}$$

We are interested in solutions taking values in a small neighborhood of a given state in $\mathbf{R^n}$ and, without loss of generality, we can choose this set to be the ball $\mathcal{U} := \mathbf{B}(\eta)$ centered at the origin with suitably small radius $\eta$. We first recall that the Riemann problem for system (1.1) is a special Cauchy problem with the piecewise constant initial data

$$t = 0 : u = \begin{cases} u_L, & x < 0, \\ u_R, & x > 0, \end{cases} \tag{1.9}$$

where $u_L$ and $u_R$ are constant states in $\mathcal{U}$. It is well-known that the Riemann problem (1.1) and (1.9) has a unique self-similar solution composed of $n+1$ constant states separated by shocks, rarefaction waves, and contact discontinuities (they are called elementary waves), provided that the states are in a small neighborhood of a given state (cf. [1]). In the following, the set $\mathcal{U}$ is chosen such that the Riemann problem is always well-posed in this sense.

We assume that on the domain under consideration, the eigenvalues of $A(u) = \nabla f(u)$ satisfy the non-characteristic condition

$$\lambda_r(u) < 0 < \lambda_s(u) \qquad (r = 1, \dots, m;\ s = m+1, \dots, n). \tag{1.10}$$

We are concerned with the global existence and uniqueness of piecewise $C^1$ solutions to the generalized nonlinear initial–boundary Riemann problem for system (1.1) in a half space

$$D = \{(t, x) \mid t \geq 0, x \geq 0\} \tag{1.11}$$

with the initial condition

$$t = 0 : u = \varphi(x)(x \geq 0) \tag{1.12}$$