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Reducing dimensionality to model 2D rotating and standing waves in a delayed nonlinear optical system with thin annulus aperture

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ABSTRACT

The purpose of the paper is to theoretically investigate 2D wave patterns in a nonlinear optical system with diffractive feedback. We consider a delayed functional differential diffusion equation on a thin annulus with Neumann boundary conditions. To study pattern formation phenomena, we construct the Faria normal form of a Hopf bifurcation, which is necessarily degenerate because the equation exhibits O(2) symmetry. The coefficients of the normal form determine the excitation of rotating or standing waves with the prescribed spatio-temporal characteristics, but these coefficients can be expressed only implicitly. However, for sufficiently thin annuli, the equation corresponds to a limit 1D model on a circle, which was carefully analysed in our previous papers and whose normal form was explicitly calculated. 2D wave stability predictions that we make based on the normal form of the limit 1D model are in good agreement with direct numerical simulations of the 2D model. To accelerate computation, we elaborated an efficient fast finite Hankel transform algorithm for thin annular domains.

1. Introduction

Nonlinear optical systems based on various configurations of Kerr-type nonlinear slices and feedback mirrors have been popular among researches since the late 1980s as an inexhaustible spring of rich spatial–temporal dynamics (see [1–5], etc. and references therein). Such systems have a number of control parameters of local and nonlocal interactions. By tuning them to his needs, an investigator can activate different spatio-temporal regimes with the prescribed characteristics. Namely, for certain values of the control parameters, the systems can solve important applied information processing problems [6] such as recognition tasks [7,8]. A different choice of parameters can lead to excitation of 2D rotating and standing waves – typical instances of self-organization phenomena – as it was observed in a series of experiments and computer simulations

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(see [1–4,9–11], etc.). In this paper, we deal with rotating and standing waves in a nonlinear optical system with thin annular aperture and propose a dimension reduction approach to study them when the model takes feedback loop delay into account.

Temporal delays appear naturally in a variety of optical systems (e.g., ring resonators with Kerr-like media) and can cause oscillatory instabilities and chaotic regimes. Such phenomena were actively discussed in the context of the Ikeda coupled differential–difference equations (DDE) [12,13]. Recently, a special asymptotic method of DDE integration was elaborated in [14] to treat relaxation oscillations in lasers. The dynamics of a laser with time-dependent delayed feedback was analysed in the neighbourhood of an equilibrium in [15].

For a spatially-distributed system, its dynamics is governed by a scalar quasilinear functional differential diffusion equation (FDDE) for phase modulation of input light wave, where the functional term corresponds to a particular type of interactions in the feedback loop. Various delay-free systems with different feedback interactions were studied in the literature: those utilizing interference of input and feedback waves with transforms of spatial arguments [4,9–11], diffraction of feedback wave [5,16], Fourier filtering [17], and their combinations [1,2,6]. In these delay-free systems, a combination of local (diffusion/diffraction) and nonlocal (rotation of spatial arguments) coupling can be used to excite rotating waves in the framework of SO(2)-symmetric FDDE. The rotation transform fixes the direction (clockwise or counterclockwise) of rotating waves and makes it possible to apply the non-degenerate Hopf bifurcation technique [18] to obtain 1D waves on a circle in the form of series expansion and study their stability [19,20]. The degenerate case of small diffusion was treated in [21,22] by means of quasi-normal forms reduction; interactions of 1D rotating waves were considered in [23,24]. 2D rotating [25–27] and spiral [9,10] waves were studied in the presence of argument rotation. The multidimensional case was investigated in [28,29].

Allowing for control signal delay makes the models of wavefront correction systems more accurate and adequate; for example, those with feedback television loop [30,31] or PC-based digitally controlled feedback [32]. Additionally, delay steps in as an extra control parameter, significantly enriching the dynamics of the system. With specially calculated parameters the system may be considered as a generator of artificial optical turbulence [33,34], being useful in various problems of optical cryptology [35]. Also, rotating waves are typical of SO(2)-symmetric delayed systems with argument rotation transformation [36,37].

In [38], for an FDDE with interference, delay, and rotation of coordinates, we assayed a two-step procedure to describe 1D rotating waves on a circle (see the 2D case in [39]). The first step consisted in passing to a rotating coordinate frame to reduce the FDDE to an elliptic functional differential equation with a shifted argument that describes the profile of a rotating wave. After that, we applied the Liapunov–Schmidt method together with the implicit operator theorem to obtain the waveform as a small parameter expansion. During the second step, on the basis of the Faria normal form reduction [40,41], we studied the stability of rotating waves by analysing the signs of the normal form coefficients.

In recent papers [42,43], we investigated a substantially more complicated case of a delayed optical system with diffraction but without spatial transformations. Here, the dynamics is governed by an O(2)-symmetric FDDE (rather than a merely SO(2)-symmetric one) and so the corresponding Hopf bifurcation is degenerate. In contrast to the SO(2)-symmetric systems, our two-step approach provided the simultaneous existence of both clockwise and counterclockwise rotating waves for the same sets of parameters. What is more, by analysing the normal form we showed that standing waves can coexist with rotating waves and obtained their stability criteria.

It should be stressed that for 1D models on a circle – both SO(2)- and O(2)-symmetric ones – the coefficients of the series expansion and of the normal form can be computed explicitly. Thus, it is possible to describe the system's spatio-temporal dynamics quantitatively and qualitatively and study stability as well. Unfortunately, explicitness is not there in 2D: namely, to calculate the coefficients of the normal form that describes 2D rotating waves on a disc, one needs to solve a number of boundary value problems for second order ordinary differential equations in the radial variable [39], which cannot be done but numerically.