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On the blow-up of solutions for the Fornberg–Whitham equation

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ABSTRACT

Wave breaking of the water wave is important and interesting to physicist and mathematician. The present article is devoted to studying wave breaking (bounded solutions with unbounded derivative) of the Fornberg–Whitham equation. By virtue of the conservation law of the L^2 norm of solutions which is found in this paper, two very useful a priori estimates are derived (see Theorem 2.1). Based on the L^2-conservation and L^∞-estimate of solutions, several blow-up phenomena of the Fornberg–Whitham equation on line $\mathbb{R}$ and on circle $\mathbb{T}$ are established.

1. Introduction

In this paper, we consider the following initial value problem of the Fornberg–Whitham (FW) equation [1]:

$$\begin{cases} \partial_t u - \partial_t u_{xx} + \dfrac{3}{2} u u_x - u_x = \dfrac{9}{2} u_x u_{xx} + \dfrac{3}{2} u u_{xxx}, \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

which was introduced by Fornberg and Whitham in 1978 as a model for describing waves breaking. Here $u = u(t,x)$ stands for the fluid velocity at time t, in the spatial x direction and the subscripts denote the partial derivatives.

As we all know, a basic question in the theory of nonlinear PDE is: when can a singularity form and what is its nature? A kind of singularity occurs if the solution itself becomes unbounded in finite time, the other singularity is that the solution remains bounded but its derivative becomes infinite in finite time. If the solution remains bounded but its slope becomes infinite in finite time, we say the wave breaks. As Whitham [2] emphasized in 1974, "the breaking phenomenon is one of the most intriguing long-standing problems of water wave theory".

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In 1895, Korteweg and De Vries derived the famous KdV equation

$$\partial_t u + uu_x + \frac{1}{6}u_{xxx} = 0$$

to study the behavior of long wave on shallow water in close agreement with the observations of Russell. It has solitary wave and a bi-Hamiltonian structure, and is completely integrable. However, because of strong dispersion effects, the KdV equation is globally in time well-posed in Sobolev space [3], for instance, in $H^1(\mathbb{R})$. As Whitham noted "it is intriguing to know what kind of simpler mathematical equation could include breaking and peaking". Since the KdV equation cannot describe wave breaking, which is observed by many physicists in experiment, he therefore suggested the following equation [2]

$$\partial_t u + uu_x + \int_{\mathbb{R}} K(x-\xi)u_x(t,\xi)d\xi = 0, \tag{1.2}$$

with the singular kernel

$$K(x) = -\frac{1}{2\pi}\int_{\mathbb{R}} \sqrt{\frac{\tanh\xi}{\xi}}e^{ix\xi}d\xi,$$

or K is other symbol of the pseudodifferential operator [1].

Applying the operator $(1-\partial_x^2)^{-1}$ on both sides of (1.1), this equation can be written in the following nonlocal form

$$\partial_t u + \frac{3}{2}uu_x = (1-\partial_x^2)^{-1}\partial_x u. \tag{1.3}$$

It is easy to check that (1.2), (1.3) belongs to the family of nonlinear wave equations

$$\partial_t u + \alpha uu_x = \mathcal{L}(u, u_x), \tag{1.4}$$

which has been studied by many authors. With $\alpha = 1$, $\mathcal{L}(u) = -\frac{1}{6}\partial_x^3$ in (1.4), it becomes the well-known KdV equation [4]. For $\alpha = 1$, $\mathcal{L}(u) = -(1-\partial_x^2)^{-1}\partial_x(u^2 + \frac{1}{2}u_x^2)$, (1.4) becomes the Camassa–Holm (CH) equation [5]

$$\partial_t u + uu_x = -(1-\partial_x^2)^{-1}\partial_x(u^2 + \frac{1}{2}u_x^2),$$

which models the unidirectional propagation of shallow water waves over a flat bottom [5] and also models the propagation of axially symmetric waves in hyperelastic rods [6]. It has a bi-Hamiltonian structure [7], a Lax pair based on a linear spectral problem of second order and is completely integrable [8]. Moreover, the CH equation has peaked solitons [9], which are orbital stable [10]. In view of three conservation laws

$$E_1(u) = \int_{\mathbb{R}} udx, \qquad E_2(u) = \int_{\mathbb{R}} (u^2 + u_x^2)dx, \qquad E_3(u) = \int_{\mathbb{R}} (u^3 + uu_x^2)dx.$$

Constantin and Escher show that the CH equation not only has global solutions but also blow-up solutions in finite time [11–13]. The CH equation is not the only integrable PDE of the following kind: being a shallow water equation whose dispersionless version has weak solitons. If $\alpha = 1$, $\mathcal{L}(u) = -\frac{3}{2}(1-\partial_x^2)^{-1}\partial_x(u^2)$, we find it becomes the Degasperis–Procesi (DP) equation [14]

$$\partial_t u + uu_x = -\frac{3}{2}(1-\partial_x^2)^{-1}\partial_x(u^2),$$

which can be regarded as a model for nonlinear shallow water dynamics [15]. Degasperis, Holm and Hone [16] prove the formal integrability of the equation by constructing a Lax pair. They also show that it has a