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Global regularity of the two-dimensional Boussinesq equations without diffusivity in bounded domains

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ABSTRACT

We address the well-posedness for the two-dimensional Boussinesq equations with zero diffusivity in bounded domains. We prove global in time regularity for rough initial data: the initial velocity has ϵ fractional derivatives in L^q and the initial temperature is in L^q, for some $q > 2$ and $\epsilon > 0$ arbitrarily small.

1. Introduction and main results

In this paper, we study the initial–boundary value problem for the 2D Boussinesq equations with zero thermal diffusivity on an open bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary $\partial\Omega$. The corresponding equations reads

$$\begin{cases} \partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p = \theta e_2, \\ \partial_t \theta + u \cdot \nabla \theta = 0, \\ \nabla \cdot u = 0, \end{cases} \tag{1.1}$$

where $u = (u_1, u_2)$ is the velocity vector field, p is the pressure, θ is the temperature, $\nu > 0$ is the constant viscosity, and $e_2 = (0, 1)$. This system is supplemented by the following initial and boundary conditions

$$\begin{cases} (u, \theta)(x, 0) = (u_0, \theta_0)(x),\ x \in \Omega, \\ u(x, t)|_{\partial\Omega} = 0. \end{cases} \tag{1.2}$$

Here, we have imposed the mostly used no-slip conditions on the velocity, which assume that fluid particles are adherent to the boundary due to the positive viscosity.

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The Boussinesq equations play an important role in modeling large scale atmospheric and oceanic flows [1,2]. In addition, the Boussinesq equations is closely related to Rayleigh–Benard convection [2]. From the mathematical view, the 2D Boussinesq equations serve as a simplified model of the 3D Euler and Navier–Stokes equations. In fact, we get the 2D Boussinesq equations when we analyze 3D axisymmetric swirling fluid in the Navier–Stokes framework. Better understanding of the 2D Boussinesq equations will undoubtedly shed light on the understanding of 3D flows [3].

Recently, the well-posedness of the 2D Boussinesq equations has attracted attention of many mathematicians, see [4–20]. In particular, when $\Omega = \mathbb{R}^2$, the Cauchy problem of (1.1) has been well studied. Hou and Li [10] and Chae [6] showed the global in time regularity for $(u_0, \theta_0) \in H^3(\mathbb{R}^2) \times H^2(\mathbb{R}^2)$. Kukavica, Wang, Ziane [18] obtained the global regularity for $(u_0, \theta_0) \in W^{1+s,q}(\mathbb{R}^2) \times W^{s,q}(\mathbb{R}^2)$ for $s \in (0, 1)$, $q \in [2, \infty)$ and $sq > 2$. They also pointed out that the restriction $sq > 2$ can be removed provided that the initial data have compact support or $\Omega = \mathbb{T}^2$. Abidi and Hmidi [4] proved the global existence for $(u_0, \theta_0) \in L^2(\mathbb{R}^2) \cap B^{-1}_{\infty,1}(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$. Danchin and Paicu [7] proved the uniqueness of weak solution for $(u_0, \theta_0) \in L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$.

In real world applications, fluids often move in bounded domains, where new phenomena such as the creation of vorticity on the boundary appears. In such case, the boundary effect requires a careful analysis. The initial–boundary value problem of (1.1)–(1.2) was first studied by Lai, Pan, and Zhao [13], who showed the global regularity for $(u_0, \theta_0) \in H^3(\Omega) \times H^2(\Omega)$. Later, Hu, Kukavica and Ziane [11] proved the global existence for initial data $(u_0, \theta_0) \in H^2(\Omega) \times H^1(\Omega)$. Recently, He [9] established the uniqueness of weak solution in the energy space $L^2(\Omega) \times L^2(\Omega)$.

In the current paper, we study further the initial–boundary value problem of system (1.1)–(1.2). We improve the previous results to the case of rough initial data $(u_0, \theta_0) \in D^{1-\frac{1}{p},p}_{A_q}(\Omega) \times W^{s,q}(\Omega)$. Here, $D^{1-\frac{1}{p},p}_{A_q}$ denotes some fractional domain of the Stokes operator whose elements have $2 - \frac{2}{p}$ derivatives in L^q (see Section 2 for the definition).

Before stating our main results, we define the function spaces in which existence is going to be shown.

Definition 1.1. For all $T > 0$, $s \geq 0$ and $1 < p, q < \infty$, we denote $M^{p,q,s}_T$ the set of triples (u, p, θ) such that

$$u \in C([0, T]; D^{1-\frac{1}{p},p}_{A_q}) \cap L^p(0, T; W^{2,q} \cap W^{1,q}_0), \partial_t u \in L^p(0, T; L^q),$$
$$p \in L^p(0, T; W^{1,q}) \text{ and } \int_\Omega p dx = 0,$$
$$\theta \in C([0, T]; W^{s,q}),\ \partial_t \theta \in L^p(0, T; W^{-1,q}).$$

The corresponding norm is denoted by $\| \cdot \|_{M^{p,q,s}_T}$.

Our main results read as follows.

Theorem 1.2. *Let Ω be a bounded domain in $\mathbb{R}^2$ with $C^{2+\epsilon}$ boundary. Let $p \in (1, \infty)$, $q \in (2, \infty)$ and $s \in [0, 1]$. Let $u_0 \in D^{1-\frac{1}{p},p}_{A_q}$ and $\theta_0 \in W^{s,q}$. Then system (1.1)–(1.2) has a unique global solution which belongs to $M^{p,q,s}_T$ for all $T > 0$.*

As a byproduct of the proof of Theorem 1.2, we get the global regularity for $(u_0, \theta_0) \in H^1(\Omega) \times H^1(\Omega)$.

Proposition 1.3. *Let Ω be a bounded domain in $\mathbb{R}^2$ with $C^{2+\epsilon}$ boundary. Assume that $u_0 \in H^1$, $\nabla \cdot u_0 = 0$ and $\theta_0 \in H^1$. Then there exists a unique global solution to system (1.1)–(1.2) such that for all $T > 0$*

$$u \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2),\ \nabla u \in L^1(0, T; L^\infty),\ \theta \in L^\infty(0, T; H^1).$$