Nonlinear Analysis: Real World Applications 43 (2018) 155–166

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

On the equivalence groups for (2+1) dimensional nonlinear diffusion equation

Saadet Özer

Istanbul Technical University, Faculty of Science and Letters, Department of Mathematics, 34469 Maslak, Istanbul, Turkey

ARTICLE INFO

Article history:
Received 16 February 2017
Received in revised form 19 February 2018
Accepted 23 February 2018

Keywords:
Lie group application
Equivalence transformations
Exact solution
Nonlinear diffusion equation

ABSTRACT

(2+1) dimensional diffusion equation is considered within the framework of the group of equivalence transformations. Generators for the group are obtained and admissible transformations between linear and nonlinear equations are examined. It is shown that transformations between linear and nonlinear equations are possible provided that the generators of independent variables depend on the dependent variable. Exact solutions for some nonlinear equations are obtained.

1. Introduction

Differential equations containing some arbitrary functions or parameters represent actually family of equations of the same structure. Almost all field equations of classical continuum physics possess this property related to the behavior of different materials. In dealing with such family of differential equations, Lie symmetry analysis provides some powerful algorithmic methods for determination of invariant solutions, conserved quantities and construction of maps between differential equations of the same family that turns out to be equivalent [1–3]. To examine such problems, it is convenient considering equivalence transformation groups that preserve the structure of the family of differential equations but may change the form of the constitutive functions, parameters when appropriate transformations are available.

The first systematic treatment that the usual Lie's infinitesimal invariance approach could be employed in order to construct equivalence groups was formulated by Ovsiannikov [4]. Then several well-developed methods have been used to construct equivalence groups. The general theory of determining transformation groups and algorithms can be found in the references [4–7].

In the present text we shall examine the $(2+1)$ dimensional diffusion equation. The nonlinear members of the family of diffusion equations have significant importance in many areas in applied sciences. A

E-mail address: saadet.ozer@itu.edu.tr.

https://doi.org/10.1016/j.nonrwa.2018.02.010

great number of members have been used to model many physical phenomena in Mathematical Physics, Mathematical Biology etc. and they have been widely studied not only by means of numerical, asymptotic analysis but also as application to Lie Group Analysis since Lie [8]. The simple nonlinear heat equation, $u_t = [A(u)u_x]_x$ was first examined by Ovsiannikov [9] within the frame-work of Lie's symmetry classification. The complete classification and form-preserving point transformations for the inhomogeneous one dimensional nonlinear diffusion equations are obtained in [10]. Conditions for reducing more general diffusion type equations to the one dimensional heat equation are also examined in [11]. Equivalence transformations of linear equations into nonlinear equations for some classes have been considered in Lisle's Ph.D. thesis [12] widely. Constructing the exact analytical solutions for some specific problems related to the nonlinear diffusion equation have recently been examined by [13–15]. Torrisi et al. in their paper [14] have applied the equivalence transformations to examine the developments of bacterial colonies. Bruzón et al. have also derived maps between nonlinear dispersive equations in their work [16]. For more details we refer the reader [17–19] and references therein which are concerned with equivalence transformations. A symbolic computation for determining equivalence transformations can also be found in the recent work [20]. We also refer the paper [21] by Tsaousi et al. in which they investigate the invariants of two dimensional linear parabolic equation.

The aim of the present work is to study equivalence transformations for a general family of $(2 + 1)$ dimensional diffusion equation. We investigate the structure of the transformation group generators which lead to map linear and nonlinear members.

For the convenience of the reader to follow, in Section 2, we have obtained the generators of the group of equivalence transformations and determined the structure for admissible transformations. Theorems in the section state the conditions of the appropriate equivalence transformations between linear and nonlinear members. In the next section, we have considered some subgroups of the general equivalence groups and by choosing some specific forms we are able to obtain some classes of nonlinear equations which are equivalent to linear ones. We have also investigated the classes of nonlinear diffusion equations that are mapped onto the classical heat equation. Exact solutions for those nonlinear equations are also obtained. Some maps between the different linear equations, such as between constant coefficient and variable coefficient equations and homogeneous to nonhomogeneous equations are also tackled.

2. Equivalence transformations

In the present paper we shall investigate the equivalence group a general family of $(2 + 1)$ dimensional diffusion equation

$$u_t = f(x, y, t, u, u_x, u_y)_x + g(x, y, t, u, u_x, u_y)_y \tag{1}$$

which represents a great variety of linear and nonlinear equations. Here u is the dependent variable of the independent variables x, y, t and f, g are smooth nonconstant functions of their variables, subscripts denote the partial derivatives with respect to the corresponding variables. Even though many members of this equation have been examined in the literature before, such general class has not been considered in any other work.

Definition 1. With n independent variables x_i, N dependent variables u_α and m smooth functions ϕ_k of independent, dependent variables and their derivatives

$$\mathcal{F}(x_i, u_{\alpha(p)}, \phi_{k(q)}(x_i, u_{\alpha(p)})) = 0$$

is called a family of differential equations. Here $i = 1, 2...n,\ \alpha = 1, 2, \ldots, N,\ k = 1, 2, \ldots, m$ and $u_{\alpha(p)}$ include both the tuple of dependent variables $u = (u_1, u_2, \ldots, u_N)$ as well as all the derivatives of u with