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An infinite dimensional Duffing-like evolution equation with linear dissipation and an asymptotically small source term

Marina Ghisi[a], Massimo Gobbino[b,*], Alain Haraux[c]

[a] Università degli Studi di Pisa, Dipartimento di Matematica, PISA, Italy
[b] Università degli Studi di Pisa, Dipartimento di Ingegneria Civile e Industriale, PISA, Italy
[c] Université Pierre et Marie Curie, Laboratoire Jacques-Louis Lions, Paris, France

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ABSTRACT

We consider an abstract nonlinear second order evolution equation, inspired by some models for damped oscillations of a beam subject to external loads or magnetic fields, and shaken by a transversal force. When there is no external force, the system has three stationary positions, two stable and one unstable, and all solutions are asymptotic for t large to one of these stationary solutions.

We show that this pattern extends to the case where the external force is bounded and small enough, in the sense that solutions can exhibit only three different asymptotic behaviors.

1. Introduction

Let H be a real Hilbert space, in which $|x|$ denotes the norm of an element $x \in H$, and $\langle x, y\rangle$ denotes the scalar product of two elements x and y. Let A be a self-adjoint positive operator on H with dense domain $D(A)$.

We consider some evolution problems of the following form

$$u'' + \delta u' + k_1 A^2 u - k_2 Au + k_3 |A^{1/2}u|^2 Au = f(t), \tag{1.1}$$

where δ, k_1, k_2, k_3 are positive constants, and $f : [0, +\infty) \to H$ is a given forcing term, with initial data

$$u(0) = u_0, \qquad u'(0) = u_1.$$

A concrete example of an equation that fits in this abstract framework is the partial differential equation

$$u_{tt} + \delta u_t + k_1 u_{xxxx} + k_2 u_{xx} - k_3 \left(\int_0^1 u_x^2 \, dx \right) u_{xx} = f(t, x) \tag{1.2}$$

* Corresponding author.
E-mail addresses: marina.ghisi@unipi.it (M. Ghisi), massimo.gobbino@unipi.it (M. Gobbino), haraux@ann.jussieu.fr (A. Haraux).

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in the strip $(t,x) \in [0,+\infty) \times [0,1]$, with boundary conditions

$$u(t,x) = u_{xx}(t,x) = 0 \qquad \forall (t,x) \in [0,+\infty) \times \{0,1\}. \tag{1.3}$$

Physical models and experiments. Eq. (1.2) appears in [1] as a model for the motion of a beam which is buckled by an external load k_2, and shaken by a transverse displacement $f(t)$ (depending only on time, in that model). Eq. (1.2) becomes a special case of (1.1) if we choose $H := L^2((0,1))$ and $Au = -u_{xx}$ with homogeneous Dirichlet boundary conditions. The "hinged ends" boundary conditions (1.3) are essential here.

A different physical model leading to equations of the form (1.2), although with different boundary conditions, is the so called magneto-elastic cantilever beam described in Figure 1 of [2]. Due to the different boundary conditions, this equation does not reduce to (1.1) but it is reasonable to expect a similar global behavior of solutions. The physical apparatus consists in a beam which is clamped vertically at the upper end, and suspended at the other end between two magnets secured to a base. The whole system is shaken by an external force transversal to the beam.

Both systems exhibit a somewhat complex behavior. To begin with, let us consider the case without external force. When k_2 is small enough, the trivial solution $u(t) \equiv 0$ is stable. This regime corresponds to a small external load in the first model, and to a large distance from the magnets in the case of the magneto-elastic beam. When k_2 increases, the trivial solution becomes unstable, and two nontrivial equilibrium states appear. In this new regime, the effect of an external force seems to depend deeply on the size of the force itself. If the force is small enough, experiments reveal that solutions remain close to the equilibrium states of the unforced system. On the contrary, when the external force is large enough, trajectories seem to show a chaotic behavior. Describing and modeling this chaotic behavior was actually the main goal of [2,1].

Simple modes and Duffing's equation. Up to changing the unknown and the operator according to the rules

$$u(t) \rightsquigarrow \alpha u(\beta t), \qquad A \rightsquigarrow \gamma A$$

for suitable values of α, β, γ, we can assume that three of the four constants in (1.1) are equal to 1. We end up, naming for simplicity the new unknown by u as well, with the equation

$$u'' + u' + A^2 u - \lambda A u + |A^{1/2}u|^2 A u = f(t) \tag{1.4}$$

with the initial conditions renamed accordingly

$$u(0) = u_0, \qquad u'(0) = u_1. \tag{1.5}$$

Just to fix ideas, we can also assume, as in the concrete example (1.2), that H admits an orthonormal basis $\{e_n\}$ made by eigenvectors of A, corresponding to an increasing sequence $\lambda_1 < \lambda_2 < \cdots$ of positive eigenvalues. If we restrict Eq. (1.4) to the kth eigenspace, we obtain an ordinary differential equation of the form

$$u_k'' + u_k' + \lambda_k(\lambda_k - \lambda)u_k + \lambda_k^2 u_k^3 = f_k(t). \tag{1.6}$$

Of course (1.4) is not equivalent to the system made by (1.6) as k varies, because of the coupling due to the nonlinear term. Nevertheless, in the special case where both initial data and the external force are multiples of a given eigenvector e_k, Eq. (1.4) reduces exactly to (1.6).

Eq. (1.6) is known in the mathematical literature as *Duffing's equation*. When there is no external force, namely $f_k(t) \equiv 0$, it is well-known that the behavior of solutions depend on the sign of the coefficient of u_k, or equivalently of $\lambda_k - \lambda$.

- When $\lambda < \lambda_k$, we are in the so-called *hardening regime*, in which the trivial solution $u_k(t) \equiv 0$ is the unique stationary solution, and it is asymptotically stable (actually all solutions tend to 0 exponentially fast as $t \to +\infty$).