Nonlinear Analysis: Real World Applications 43 (2018) 495–514

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

The boundary focus–saddle bifurcation in planar piecewise linear systems. Application to the analysis of memristor oscillators

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A R T I C L E I N F O

Article history:
Received 21 February 2017
Accepted 28 March 2018

Keywords:
Qualitative theory of odes
Piecewise linear systems
Limit cycles
Bifurcations

A B S T R A C T

Among the boundary equilibrium bifurcations in planar continuous piecewise linear systems with two zones separated by a straight line, the focus–saddle bifurcation corresponds with a one-parameter transition from a situation without equilibria to a configuration with two equilibria, namely a focus and a saddle point. Depending on the dynamics of the two linear systems involved, the focus can appear surrounded by a limit cycle, by a saddle-loop (homoclinic connection) or by nothing else.

After introducing a criticality coefficient whose sign discriminates the different possible situations, the focus–saddle bifurcation is quantitatively characterized for the first time. The analysis requires to work in a more general framework, as is the family of planar refracting linear systems with two zones, for which a new result about existence and uniqueness of limit cycles and saddle-loops is also shown.

The achieved results are applied to the study of oscillations in an electronic circuit involving a single memristor cell, showing rigorously the appearance of limit cycles via the focus–saddle bifurcation analyzed in the paper.

1. Introduction and statement of main results

In this paper, we consider continuous piecewise linear differential systems (PWL systems, for short) with two linearity zones separated by a straight line representing the only non-smoothness manifold in the phase plane. Such systems are relevant in different application fields and have been studied by several authors; see [1,2] and references therein for a thorough introduction and a revision of the state of the art in the study of these systems.

Our main goal is to characterize in a precise way the focus–saddle boundary equilibrium bifurcation for such continuous PWL systems, a phenomenon leading to the creation/annihilation of two equilibrium points (a focus and a saddle) along with the possible appearance of a limit cycle or a saddle connection surrounding the focus, see Chapter 5 of [1] and [3,4].

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https://doi.org/10.1016/j.nonrwa.2018.03.011
1468-1218/

We will assume without loss of generality that the linearity regions in the phase plane are the left and right half-planes,

$$S_L = \{(x, y) \in \mathbb{R}^2 : x < 0\}, \quad S_R = \{(x, y) \in \mathbb{R}^2 : x > 0\},$$

separated by the straight line $\Sigma = \{(x, y) \in \mathbb{R}^2 : x = 0\}$.

We proceed by considering a specific family of systems within the more general setting of discontinuous systems. Namely, we first study discontinuous PWL systems for which the two involved linear vector fields share their normal component to Σ at all the points of such discontinuity manifold. These systems are known as *refracting systems* (see [5]) and are characterized by not having any sliding segment in the discontinuity manifold Σ. In other words, orbits arriving at Σ can be naturally extended by concatenating solutions from both sides, since all the orbits hit Σ in the so-called *sewing* or crossing points, see [6]. In fact, refracting systems belong to the slightly wider class of *sewing systems*, characterized by sharing only the sign of the normal component to the discontinuity manifold Σ, see [7].

To reduce the number of parameters in the analysis, we start by introducing a canonical form for Σ-refracting PWL systems. A general discontinuous PWL system can be written as

$$\dot{\mathbf{x}} = \begin{cases} A_R\mathbf{x} + \mathbf{b}_R, & \text{if } x \in S_R, \\ A_L\mathbf{x} + \mathbf{b}_L, & \text{if } x \in S_L \cup \Sigma, \end{cases} \tag{1}$$

where $\mathbf{x} = (x, y) \in \mathbb{R}^2$ is the vector of state variables, $A_R = (a_{ij}^R)$ and $A_L = (a_{ij}^L)$ are 2×2 constant matrices, $\mathbf{b}_R = (b_1^R, b_2^R)^T$, $\mathbf{b}_L = (b_1^L, b_2^L)^T \in \mathbb{R}^2$ are constant vectors, and the point denotes derivatives with respect to the time variable s. Imposing now the refracting condition on these systems, we require that for all $\mathbf{x} = (0, y)^T \in \Sigma$, with $y \in \mathbb{R}$,

$$\mathbf{e}_1^T (A_R\mathbf{x} + \mathbf{b}_R) = \mathbf{e}_1^T (A_L\mathbf{x} + \mathbf{b}_L), \tag{2}$$

where $\mathbf{e}_1^T$ stands for the first unitary vector in row form. From (2) and some elementary algebra, we must assume $a_{12}^L = a_{12}^R$ and $b_1^L = b_1^R$, so that systems to be studied become

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{pmatrix} a_{11}^L & a_{12} \\ a_{21}^L & a_{22}^L \end{pmatrix} \mathbf{x} + \begin{pmatrix} b_1 \\ b_2^L \end{pmatrix} \text{ if } \mathbf{x} \in S_L \cup \Sigma, \\ \dot{\mathbf{x}} &= \begin{pmatrix} a_{11}^R & a_{12} \\ a_{21}^R & a_{22}^R \end{pmatrix} \mathbf{x} + \begin{pmatrix} b_1 \\ b_2^R \end{pmatrix} \text{ if } \mathbf{x} \in S_R, \end{aligned} \tag{3}$$

where $a_{12} = a_{12}^L = a_{12}^R$ and $b_1 = b_1^L = b_1^R$.

Remark 1. Note that it is natural to assume $a_{12} \neq 0$ in (3) if one wants to cope with non-elementary dynamics. Otherwise, we should have $\dot{x} = b_1$ for all the points in Σ, so that all the orbits would cross Σ in the same direction. Furthermore, the dynamics of the first variable would be decoupled from the second one in the whole plane, oscillations would be not possible and the system could not exhibit a proper two-dimensional dynamics.

As done in Proposition 3.1 of [8], by assuming $a_{12} \neq 0$ and applying the homeomorphism $\widetilde{\mathbf{x}} = h(\mathbf{x})$ given by

$$\begin{aligned} \widetilde{\mathbf{x}} &= \begin{pmatrix} 1 & 0 \\ a_{22}^L & -a_{12} \end{pmatrix} \mathbf{x} - \begin{pmatrix} 0 \\ b_1 \end{pmatrix} \text{ if } \mathbf{x} \in S_L \cup \Sigma, \\ \widetilde{\mathbf{x}} &= \begin{pmatrix} 1 & 0 \\ a_{22}^R & -a_{12} \end{pmatrix} \mathbf{x} - \begin{pmatrix} 0 \\ b_1 \end{pmatrix} \text{ if } \mathbf{x} \in S_R, \end{aligned}$$