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# Global classical solutions of compressible isentropic Navier–Stokes equations with small density

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ABSTRACT

This paper concerns the Cauchy problem of compressible isentropic Navier–Stokes equations in the whole space $\mathbb{R}^3$. First, we show that if $\rho_0 \in L^\gamma \cap H^3$, then the problem has a unique global classical solution on $\mathbb{R}^3 \times [0, T]$ with any $T \in (0, \infty)$, provided the upper bound of the initial density is suitably small and the adiabatic exponent $\gamma \in (1, 6)$. If, in addition, the conservation law of the total mass is satisfied (i.e., $\rho_0 \in L^1$), then the global existence theorem with small density holds for any $\gamma > 1$. It is worth mentioning that the initial total energy can be arbitrarily large and the initial vacuum is allowed. Thus, the results obtained particularly extend the one due to Huang–Li–Xin (Huang et al., 2012), where the global well-posedness of classical solutions with small energy was proved.



## 1. Introduction

The motion of viscous isentropic compressible fluids occupying a domain $\Omega \subset \mathbb{R}^3$ is governed by the Navier–Stokes equations:

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \\ (\rho u)_t + \mathrm{div}(\rho u \otimes u) + \nabla P(\rho) = \mu \Delta u + (\mu + \lambda)\nabla \mathrm{div} u, \end{cases} \tag{1.1}$$

where $\rho \geq 0$ and $u = (u^1, u^2, u^3)$ are the fluid density and velocity, respectively. The pressure $P = P(\rho)$ is determined through the so-called $\gamma$-law:

$$P(\rho) = A\rho^\gamma \quad \text{with} \quad A > 0,\ \gamma > 1, \tag{1.2}$$

where $\gamma > 1$ is the adiabatic exponent and $A > 0$ is a physical constant. The viscosity coefficients $\mu$ and $\lambda$ satisfy the physical restrictions:

$$\mu > 0, \quad 2\mu + 3\lambda \geq 0. \tag{1.3}$$

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Let $\Omega=\mathbb{R}^3$. We aim to look for the solutions, $(\rho,u)(x,t)$, to the Cauchy problem of (1.1) with the far-field behavior:

$$(\rho,u)(x,t)\to(0,0)\quad\text{as}\quad |x|\to\infty, \tag{1.4}$$

and the initial data

$$(\rho,u)(x,0)=(\rho_0,u_0)(x)\quad\text{with}\quad x\in\mathbb{R}^3. \tag{1.5}$$

A great number of works have been devoted to the well-posedess theory of the multi-dimensional compressible Navier–Stokes equations. Local existence and uniqueness of classical solutions was studied in [1,2] and [3–6] in the non-vacuum and vacuum case, respectively. In [7], Matsumura–Nishida first proved the global existence of smooth solutions with initial data close to a non-vacuum equilibrium in $H^3$-norm. Later, Hoff [8,9] extended Matsumura–Nishida's result to the case of discontinuous initial data. The major breakthrough of global existence with large data is due to Lions [10] (see also Feireisl et al. [11]), where the author proved the global existence of weak solutions, the so-called finite-energy weak solutions, when the adiabatic exponent $\gamma$ is suitably large (i.e., $\gamma>3/2$). It is worth mentioning that the initial energy was only assumed to be finite in [10,11], so that, the initial data may be arbitrary large and the density may vanish or even has compact support. However, the uniqueness and regularity of such weak solutions is still unknown. Recently, Huang–Li–Xin [12] established the global well-posedness of classical solutions with large oscillations and vacuum of (1.1)–(1.5) under the assumption that the initial energy is suitably small.

In [13,14], the authors proved independently that if the shear viscosity coefficient $\mu$ is suitably large such that $\mu\geq\lambda/7$, then the upper bound of the density dominates the blowup mechanism of strong solutions. In other words, the problem (1.1)–(1.5) has a unique global strong solution on $\mathbb{R}^3\times[0,T]$, provided the density remains bounded on $\mathbb{R}^3\times[0,T]$. Motivated by [13,14] and based on the ideas in [12], Deng–Zhang–Zhao [15] proved the global classical solutions of (1.1)–(1.5) with large data in the case when $\mu>0$ is large enough. Very recently, Zhang–Zhang–Zhao [16] obtained the global well-posedness of classical solutions, under the assumptions that the upper bound of the density is suitably small and the adiabatic exponent $\gamma\in(1,3/2)$.

The main purpose of this paper is to improve the result obtained in [16], and to exclude the unsatisfactory restriction on the adiabatic exponent (i.e., $\gamma\in(1,3/2)$). We shall work with the standard homogeneous and inhomogeneous Sobolev spaces:

$$\begin{cases} L^r\triangleq L^r(\mathbb{R}^3),\quad D^{k,r}\triangleq\{u\in L^1_{\text{loc}}\mid \|\nabla^k u\|_{L^r}<\infty\},\quad \|u\|_{D^{k,r}}\triangleq\|\nabla^k u\|_{L^r},\\ W^{k,r}\triangleq L^r\cap D^{k,r},\quad H^k=W^{k,2},\quad D^k\triangleq D^{k,2},\ D^1=\{u\in L^6\mid \|\nabla u\|_{L^2}<\infty\},\end{cases}$$

where $k\in\mathbb{Z}$ and $1<r<\infty$. The total energy is defined as follows:

$$E(t)\triangleq\int\left(\frac12\rho|u|^2+\frac{A}{\gamma-1}\rho^\gamma\right)(x,t)dx\quad\text{with}\quad\int f(x)dx\triangleq\int_{\mathbb{R}^3}f(x)dx, \tag{1.6}$$

and the initial energy is denoted by $E_0$, i.e.,

$$E_0\triangleq E(0)=\int\left(\frac12\rho_0|u_0|^2+\frac{A}{\gamma-1}\rho_0^\gamma\right)(x)dx. \tag{1.7}$$

The first result of this paper is formulated in the following theorem.

**Theorem 1.1.** *For any given positive numbers $M_0,M_1$ and $M_2$, suppose that*

$$\begin{cases}\rho_0|u_0|^2+\rho_0^\gamma\in L^1,\quad u_0\in D^1\cap D^3,\quad(\rho_0,P(\rho_0))\in H^3,\\ 0\leq\inf\rho_0\leq\rho_0(x,t)\leq\sup\rho_0\leq M_0,\quad \|\nabla u_0\|^2_{L^2}\leq M_1,\end{cases} \tag{1.8}$$