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Existence of traveling wave solutions of parabolic–parabolic chemotaxis systems

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ABSTRACT

The current paper is devoted to the study of traveling wave solutions of the following parabolic–parabolic chemotaxis system,

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + u(a - bu), \quad x \in \mathbb{R}^N \\ \tau v_t = \Delta v - v + u, \quad x \in \mathbb{R}^N, \end{cases}$$

where $u(x,t)$ represents the population density of a mobile species and $v(x,t)$ represents the population density of a chemoattractant, and χ represents the chemotaxis sensitivity.

In an earlier work (Rachidi et al., 2017) by the authors of the current paper, traveling wave solutions of the above chemotaxis system with $\tau = 0$ are studied. It is shown in Rachidi et al. (2017) that for every $0 < \chi < \frac{b}{2}$, there is $c^*(\chi)$ such that for every $c > c^*(\chi)$ and $\xi \in S^{N-1}$, the system has a traveling wave solution $(u(x,t), v(x,t)) = (U(x \cdot \xi - ct; \tau), V(x \cdot \xi - ct; \tau))$ with speed c connecting the constant solutions $(\frac{a}{b}, \frac{a}{b})$ and $(0,0)$. Moreover,

$$\lim_{\chi \to 0+} c^*(\chi) = \begin{cases} 2\sqrt{a} & \text{if} \quad 0 < a \leq 1 \\ 1 + a & \text{if} \quad a > 1. \end{cases}$$

We prove in the current paper that for every $\tau > 0$, there is$0 < \chi_\tau^* < \frac{b}{2}$ such that for every $0 < \chi < \chi_\tau^*$, there exist two positive numbers $c^{**}(\chi,\tau) > c^*(\chi,\tau) \geq 2\sqrt{a}$ satisfying that for every $c \in (c^*(\chi,\tau)\ ,\ c^{**}(\chi,\tau))$ and $\xi \in S^{N-1}$, the system has a traveling wave solution $(u(x,t), v(x,t)) = (U(x \cdot \xi - ct; \tau), V(x \cdot \xi - ct; \tau))$ with speed c connecting the constant solutions $(\frac{a}{b}, \frac{a}{b})$ and $(0,0)$, and it does not have such traveling wave solutions of speed less than $2\sqrt{a}$. Moreover,

$$\lim_{\chi \to 0+} c^{**}(\chi,\tau) = \infty,$$

$$\lim_{\chi \to 0+} c^*(\chi,\tau) = \begin{cases} 2\sqrt{a} & \text{if} \quad 0 < a \leq \dfrac{1+\tau a}{(1-\tau)_+} \\ \dfrac{1+\tau a}{(1-\tau)_+} + \dfrac{a(1-\tau)_+}{1+\tau a} & \text{if} \quad a \geq \dfrac{1+\tau a}{(1-\tau)_+}, \end{cases}$$

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and

$$\lim_{x\to\infty} \frac{U(x;\tau)}{e^{-\mu x}} = 1,$$

where μ is the only solution of the equation $\mu + \frac{a}{\mu} = c$ in the interval $(0\ ,\ \min\{\sqrt{a}, \sqrt{\frac{1+\tau a}{(1-\tau)_+}}\})$. Furthermore,

$$\lim_{\tau\to 0+} \chi^*_\tau = \frac{b}{2}, \quad \lim_{\tau\to 0+} c^*(\chi;\tau) = c^*(\chi), \quad \lim_{\tau\to 0+} c^{**}(\chi;\tau) = \infty.$$

1. Introduction

At the beginning of 1970s, Keller and Segel (see [1,2]) introduced systems of partial differential equations of the following form to model the time evolution of the density $u(x,t)$ of a mobile species and the density $v(x,t)$ of a chemoattractant,

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u - \chi(u,v)\nabla v) + f(u,v), & x\in\Omega \\ \tau v_t = \Delta v + g(u,v), & x\in\Omega \end{cases} \tag{1.1}$$

complemented with certain boundary condition on $\partial\Omega$ if Ω is bounded, where $\Omega\subset\mathbb{R}^N$ is an open domain; $\tau \geq 0$ is a non-negative constant linked to the speed of diffusion of the chemical; the function $\chi(u,v)$ represents the sensitivity with respect to chemotaxis; and the functions f and g model the growth of the mobile species and the chemoattractant, respectively. In literature, (1.1) is called the Keller–Segel model or a chemotaxis model.

Since the works by Keller and Segel, a large amount of research has been carried out toward various dynamical aspects of (1.1) on bounded domain, including global existence and finite time blow up of classical solutions, large time behavior of bounded global solutions, pattern formation, etc. It is seen that chemotactic cross-diffusion may induce rich dynamics in (1.1). For example, in (1.1) with $D(u)\equiv 1$, $\chi(u,v) = u$, $f(u,v) = 0$, and $g(u,v) = -v+u$, finite time blow-up might occur (see [3–5] for $\tau = 0$ and [6] for $\tau = 1$). When $D(u)\equiv 1$, $\chi(u,v) = u$, $g(u,v) = -v+u$, and $f(u,v)$ is a logistic source function, that is, $f(u,v) = u(a-bu)$ with $a > 0$ and $b > 0$, the blow-up phenomena may be suppressed to some extend (see [7] for $\tau = 0$ and [8] for $\tau = 1$).

Consider (1.1) on the unbounded domain $\Omega = \mathbb{R}^N$. In addition to those important dynamical issues for chemotaxis models on bounded domains, traveling wave solutions are also among important solutions of chemotaxis models on $\mathbb{R}^N$. Several researchers have studied these solutions for the choice of $D(u)\equiv 1$, $\tau = 1$, $\chi(u,v) = \chi\log(v)$, and $g(u,v) = -uv^m (0\leq m\leq 1)$ and f being of different types. For these choices of $D(u)$, $\chi(u,v)$, and $g(u,v)$ with $f(u)\equiv 0$ and $0 < m < 1$, Keller and Segel [9], showed that (1.1) can reproduce the traveling bands whose speeds were in satisfactory agreement with experimental observations. While for $m = 1$ and $f(u) = \kappa u(1-u)$, [10] proved the existence of traveling wave solutions of (1.1). There are many studies on traveling wave solutions of several other types of chemotaxis models, see, for example, [10–16], etc. In particular, the reader is referred to the review paper [16]. It should be pointed out that no much is yet known about the stability of these traveling wave solutions. None of the above mentioned works consider (1.1) on $\mathbb{R}^N$ with $D(u)\equiv 1$, $\chi(u,v) = u$, $g(u,v) = -v+u$, and $f(u,v) = u(a-bu)$, that is,

$$\begin{cases} u_t = \Delta u - \chi\nabla(u\nabla v) + u(a-bu), & x\in\mathbb{R}^N \\ \tau v_t = \Delta v - v + u, & x\in\mathbb{R}^N. \end{cases} \tag{1.2}$$