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Existence of solutions of integral equations with asymptotic conditions[☆]

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ABSTRACT

In this work we will consider integral equations defined on the whole real line and look for solutions which satisfy some certain kind of asymptotic behavior. To do that, we will define a suitable Banach space which, to the best of our knowledge, has never been used before. In order to obtain fixed points of the integral operator, we will consider the fixed point index theory and apply it to this new Banach space.

1. Introduction

In this paper we study the existence of fixed points of integral operators of the form

$$Tu(t) = p(t) + \int_{-\infty}^{\infty} k(t,s)\,\eta(s)\,f(s,u(s))\,\mathrm{d}\,s.$$

There are many results in the recent literature in which the authors deal with differential or integral problems in unbounded intervals (see for instance [1–5] and the references therein). The main difficulties which appear while dealing with this kind of problems arise as a consequence of the lack of compactness of the operator. In all of the cited references the authors solve this problem by means of the following relatively compactness criterion (see [6,7]) which involves some stability condition at $\pm\infty$:

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Theorem 1.1 *([7, Theorem 1]). Let E be a Banach space and $\mathscr{C}(\mathbb{R},E)$ the space of all bounded continuous functions $x:\mathbb{R}\to E$. For a set $D\subset\mathscr{C}(\mathbb{R},E)$ to be relatively compact, it is necessary and sufficient that:*

1. *$\{x(t),\ x\in D\}$ is relatively compact in E for any $t\in\mathbb{R}$;*
2. *for each $a>0$, the family $D_a:=\{x|_{[-a,a]},\ x\in D\}$ is equicontinuous;*
3. *D is stable at $\pm\infty$, that is, for any $\varepsilon>0$, there exists $T>0$ and $\delta>0$ such that if $\|x(T)-y(T)\|\le\delta$, then $\|x(t)-y(t)\|\le\varepsilon$ for $t\ge T$ and if $\|x(-T)-y(-T)\|\le\delta$, then $\|x(t)-y(t)\|\le\varepsilon$ for $t\le -T$, where x and y are arbitrary functions in D.*

By using the previous result, the authors of the aforementioned references prove the existence of solutions of differential or integral problems by means of either Schauder's fixed point theorem or lower and upper functions method.

In this paper, we will deal with the problem of compactness of the integral operator using a different strategy: we will define a suitable Banach space, which will be proved to be isometric isomorphic to the space

$$\mathscr{C}^n(\overline{\mathbb{R}},\mathbb{R}):=\left\{f:\overline{\mathbb{R}}\to\mathbb{R}\ :\ f|_{\mathbb{R}}\in\mathscr{C}^n(\mathbb{R},\mathbb{R}),\ \exists\lim_{t\to\pm\infty}f^{(j)}(t)\in\mathbb{R},\ j=0,\dots,n\right\}.$$

This isomorphism will allow us to apply Arzelà –Ascoli's Theorem to our Banach space instead of using Theorem 1.1.

Moreover, the Banach space that we will define will include some asymptotic condition which will ensure a certain asymptotic behavior of the solutions of the problem. Later on, we will use index theory in general cones [8] to obtain the desired fixed points.

The paper is divided in the following way: in Section 2 we present a physical problem which motivates the importance of the asymptotic behavior of solutions of a differential equation. In Section 3 we first summarize classical definitions of asymptotic behavior and then define a suitable Banach space and study its properties. Section 4 includes results of existence of fixed points of integral equations by means of the theory of fixed point index in cones. Finally, in Section 5 we will reconsider the physical problem presented in Section 2 and we will solve it by using the results given in Section 4.

2. Motivation

In many contexts it is interesting to anticipate the asymptotic behavior of the solution of a differential problem. For instance, consider the classical projectile equation that describes the motion of an object that is launched vertically from the surface of a planet towards deep space [9],

$$u''(t)=-\frac{g\,R^2}{(u(t)+R)^2},\ t\in[0,\infty);\ u(0)=0,\ u'(0)=v_0, \tag{2.1}$$

where u is the distance from the surface of the planet, R is the radius of the planet, g is the surface gravity constant and v_0 the initial velocity. Clearly, if v_0 is not big enough, the projectile will reach a maximum height, at which u' will be zero, and then fall. Hence, in order to compute the minimum velocity necessary for the projectile to escape the planet's gravity, it is enough to consider that $u(t)\to\infty$ and $u'(t)\to 0$. Then, multiplying both sides of (2.1) by u' and integrating between 0 and t,

$$\frac{1}{2}[(u'(t))^2-v_0^2]=g\,R^2\left[\frac{1}{R+u(t)}-\frac{1}{R}\right].$$

Thus, taking the limit when $t\to\infty$, $-v_0^2/2=-gR$, that is, the escape velocity is $v_s=\sqrt{2gR}$. Observe that, with $v_0=v_s$, we have

$$u'(t)=\sqrt{\frac{2gR^2}{u(t)+R}}.$$