Nonlinear Analysis: Real World Applications 42 (2018) 207–219

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

Multiple nodal solutions for nonlinear nonhomogeneous elliptic problems with a superlinear reaction ☆

Tieshan He*, Pengfei Guo, Yehui Huang, Youfa Lei

School of Computational Science, Zhongkai University of Agriculture and Engineering, Guangzhou 510225, PR China

A R T I C L E I N F O

Article history:
Received 14 September 2017
Received in revised form 27 December 2017
Accepted 29 December 2017

Keywords:
Nonhomogeneous differential operator
Nodal solutions
Variational approach
Gradient flow
Superlinear reaction

A B S T R A C T

We consider nonlinear Dirichlet and Neumann problems driven by a nonlinear nonhomogeneous differential operator and with a $(p-1)$-superlinear Carathéodory reaction which does not satisfy the Ambrosetti–Rabinowitz condition. We prove two multiplicity theorems producing two nodal solutions, and answer two open questions posed by Aizicovici et al. (2013) and by Barletta and Papageorgiou (2014). Our approach uses variational methods together with suitable truncation techniques and flow invariance arguments.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$ and let $1 < p < \infty$. In this paper we study the nonlinear nonhomogeneous Dirichlet boundary value problem

$$-\operatorname{div} a(z, Du(z)) = f(z, u(z)) \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = 0, \tag{1}$$

and the nonlinear nonhomogeneous Neumann boundary value problem

$$-\operatorname{div} a(z, Du(z)) = f(z, u(z)) \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n}|_{\partial\Omega} = 0, \tag{2}$$

where n denotes the outward unit normal vector on $\partial\Omega$, and $a : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous map such that, for every $z \in \overline{\Omega}$, $a(z, \cdot)$ is strictly monotone on $\mathbb{R}^N$, while $(z, y) \to a(z, y)$ is C^1 on $\overline{\Omega} \times (\mathbb{R}^N \setminus \{0\})$. Also,

☆ Supported by Science and technology plan foundation of Guangzhou (No. 201607010218) and by Department of Education of Guangdong Provincial Youth Innovative Talent Project (KA1705082).

* Corresponding author.
E-mail address: hetieshan68@163.com (T. He).

https://doi.org/10.1016/j.nonrwa.2017.12.010
1468-1218/

the reaction term $f(z,x) : \Omega \times \mathbb{R} \to \mathbb{R}$ is assumed to be a Carathéodory function (that is, for all $x \in \mathbb{R}$, the function $z \to f(z,x)$ is measurable and for almost all $z \in \Omega$, the function $x \to f(z,x)$ is continuous), which exhibits $(p-1)$-superlinear growth near $\pm\infty$ but without satisfying the usual (in such cases) Ambrosetti–Rabinowitz condition (the AR-condition for short). Instead, we employ an alternative weaker condition, which incorporates in our framework functions with "slower" growth near $\pm\infty$. In addition, we assume that the reaction term $f(z,x)$ admits zeros of constant sign. Our goal is to prove two multiplicity theorems producing two nodal solutions for such problems.

Recently, multiplicity results were proved for superlinear equations driven by nonhomogeneous differential operators. We refer to the works of Aizicovici–Papageorgiou–Staicu [1], Papageorgiou–Rocha [2], Papageorgiou–Winkert [3] (Dirichlet problems), Aizicovici–Papageorgiou–Staicu [4], Barletta–Papageorgiou [5], Hu–Papageorgiou [6] (Neumann problems), Hu–Papageorgiou [7] (Dirichlet and Neumann problems) and Papageorgiou–Rădulsecu [8], Papageorgiou–Winkert [9] (Robin problems). None of the works in [1,2,5,8] produces nodal solutions and instead they generate at most five nontrivial solutions. The authors in [3,4,6,7,9] assume that the reaction term $f(z,\cdot)$ is concave (i.e., $(p-1)$-sublinear) near zero, and produce a nodal solution by establishing two extremal constant sign solutions. In the particular case of the Dirichlet $(p,2)$-equations, using Morse theory and strengthening the regularity of $f(z,\cdot)$, namely, assuming that $f(z,\cdot) \in C^1(\mathbb{R})$ and

$$f'_x(z,0) \in [\hat{\lambda}_m, \hat{\lambda}_{m+1}], \quad f'_x(z,0) \neq \hat{\lambda}_m, \quad f'_x(z,0) \neq \hat{\lambda}_{m+1}$$

for some $m \geq 2$, with $\hat{\lambda}_m$ being the mth eigenvalue of $(-\Delta, H_0^1(\Omega))$, the authors in [3,7] generate a second nodal solution.

Our work is closely related to the papers of Aizicovici–Papageorgiou–Staicu [1] and Barletta–Papageorgiou [5] and in fact it complements them. More precisely, the authors in [1,5] produced five nontrivial solutions, four of constant sign, but they were unable to determine the sign of the fifth solution. Here, by using the two constant sign solutions obtained in [1, Theorem 2; 6, Theorem 4.1], we show that the fifth solution is nodal. Moreover, we derive the sixth solution being nodal for problems (1) and (2). Our results give an answer to two open questions raised by Aizicovici–Papageorgiou–Staicu [1, Remarks, p. 173] and Barletta–Papageorgiou [5, Remark 4.1, p. 911].

We stress that our results are proved without assuming any differentiability and concavity near zero on $f(z,\cdot)$ (see hypotheses $H_1(f)$ and $H_2(f)$ below). Our approach is variational based on the critical point theory coupled with suitable truncation techniques and flow invariance arguments. The main mathematical tools which will be used in this paper are recalled in the next section for the convenience of the reader. We also present the hypotheses on the map $a(\cdot,\cdot)$ and state some useful consequences of them.

2. Mathematical background-hypotheses

In the analysis of problems (1) and (2), we will use the Sobolev spaces $W_0^{1,p}(\Omega)$ and $W^{1,p}(\Omega)$, respectively. By $\|\cdot\|$ we denote the norm of the Sobolev space $W^{1,p}(\Omega)$ defined by

$$\|u\| = \left(\|u\|_p^p + \|Du\|_p^p\right)^{\frac{1}{p}} \quad \text{for all } u \in W^{1,p}(\Omega),$$

where $\|\cdot\|_s$ stands for the norm in $L^s(\Omega)$ $(1 \leq s \leq +\infty)$. Also, by $\|\cdot\|$ we denote the norm of the Sobolev space $W_0^{1,p}(\Omega)$, that is

$$\|u\| = \|Du\|_p \quad \text{for all } u \in W_0^{1,p}(\Omega)$$

(by virtue of the Poincaré inequality). However, no confusion is possible, since it will be clear from the context which norm is used. By $|\cdot|$ we denote the norm on $\mathbb{R}^N$, by $(\cdot,\cdot)_{\mathbb{R}^N}$ the inner product in $\mathbb{R}^N$ and by $|\cdot|_N$ the Lebesgue measure on $\mathbb{R}^N$. We will also use the following Banach spaces:

$$C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\} \quad \text{and} \quad C^1(\overline{\Omega}).$$