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Existence of global solutions and attractors for the parabolic equation with critical Sobolev and Hardy exponent in $\mathbb{R}^N$

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ABSTRACT

We consider asymptotic behavior for a parabolic equation (*p-heat equation*) with (sub-)critical Sobolev and Hardy exponent potential

$$u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) + f(u) = V(x)\frac{|u|^{q-2}}{|x|^s}u,$$

where $1 < p < N, (N \geq 2)$. If $V(x) \equiv \lambda$ and $s = q = p$, it is shown by Azorero and Alonso (1998) that the solution for such Cauchy problem depends strongly on p and the best constant $\lambda_{N,p}$ in Hardy's inequality, specifically, the local or global solutions for this problem depend on the different cases of $1 < p < 2, p \geq 2$ and $\lambda > \lambda_{N,p}, \lambda < \lambda_{N,p}$. By using the standard domain expansion technique and some convenient integrability conditions on the weighted function $V(x)$, we first establish that the Cauchy problem has at least one global solution for every $1 < p < N$. We then apply the theory of multivalued semigroups or multivalued semiflows to get $L^2(\mathbb{R}^N)$ global attractor for the *p-heat equation* with $2 \leq p < N$. Furthermore, the global attractor also belongs to $L^2(\mathbb{R}^N) \cap L^\alpha(\mathbb{R}^N)$ when N is suitably large.

1. Introduction

The existence of solutions for the parabolic equations with a subcritical or critical singular potential attracted much attention for many years. García Azorero and Peral Alonso [1] studied the nonlinear critical *p-heat equation*

$$\begin{cases} u_t - \Delta_p u = \dfrac{\lambda}{|x|^p}|u|^{p-2}u, & x \in \Omega,\ t > 0,\ \lambda \in \mathbb{R}, \\ u(0,x) = u_0, & x \in \Omega. \\ u(t,x) = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{1.1}$$

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where $-\Delta_p u \equiv \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $u_0 \in L^2(\Omega)$ verifying convenient regularity assumptions, Ω is a bounded domain in $\mathbb{R}^N$ such that $0 \in \Omega$ and $1 < p < N$. It is proved that the results of (1.1) depend in general on the relationship between λ and $\lambda_{N,p}$, where $\lambda_{N,p} = \frac{1}{C_{N,p}} = ((N-p)/p)^p$ (see Theorem 2.6) is the best constant in the corresponding Hardy's inequality in $W^{1,p}(\mathbb{R}^N)$. In specifically, it shows that

(1) If $p > 2, 0 < \lambda < \lambda_{N,p}$, then (1.1) has a global solution;
(2) If $p > 2, \lambda > \lambda_{N,p}$, $u_0 \in L^\infty(\Omega), u_0 \geq 0$ and $u_0 > \delta > 0$ is a neighborhood of the original, then (1.1) has no local solution;
(3) If $1 < p < 2N/(N+2), \lambda > \lambda_{N,p}, u_0(x) > 0$, then (1.1) has a global solution;
(4) If $2N/(N+2) < p < 2N/(N+1), \lambda > \lambda_{N,p}$, $u_0 \in L^\infty(\Omega), u_0 \geq 0$ then (1.1) has a global solution in the sense of distributions.

When $p = 2$ in (1.1), we also can find the similar results as above in [2] in terms of the value of λ. For other interesting results for degenerate parabolic equations, we refer to [3,4]. The variational methods are usually used to study the existence and multiplicity of solutions for the elliptic problem. Ghoussoub and Yuan [5] considered the following quasi-linear partial differential equation involving critical Sobolev and Hardy exponent

$$\begin{cases} -\Delta_p u = \mu |u|^{r-2}u + \lambda \dfrac{|u|^{q-2}}{|x|^s}u, \ x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.2}$$

where λ and μ are two positive parameters. In order to get some variational framework for the energy-functional of (1.2), authors in [5] assume that

$$1 < p < N, \ p \leq q \leq p^*(s) \equiv \frac{N-s}{N-p}p, \ p \leq r \leq p* \equiv \frac{Np}{N-p}. \tag{1.3}$$

When we consider the elliptic problems with singular potential $1/|x|^s$, we often refer to the following Sobolve–Hardy inequality, which is essentially due to Caffarelli, Kohn and Nirenberg [6]. Assume that $1 < p < N$ and that $q \leq p_*(s)$, then there is a constant $C_{N,p,q,s} > 0$ such that

$$\left(\int_{\mathbb{R}^N} \frac{|u|^q}{|x|^s} \mathrm{d}x\right)^{\frac{p}{q}} \leq C_{N,p,q,s} \int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x \text{ for all } u \in H_0^{1,p}(\mathbb{R}^N). \tag{1.4}$$

It is obvious that the singular potential in (1.2) is reduced to the one in (1.1) when $s = p$. Eq. (1.1) becomes the heat equation with inverse-square potentials and (1.2) reduces to the corresponding elliptic equation, respectively, when $p = 2$. The elliptic version (without u_t) with singular potentials comes from some physical models, such as the Schrödinger equation in quantum mechanics (see [7]), the linearized analysis of standard combustion models (see [8–10] and so on). As it is pointed out in [1], Eqs. (1.1), (1.2) provide some new phenomena when $p \neq 2$, and the evolutionary p-Laplacian equation (without singular potential) usually arise in geometry and non-Newtonian fluid mechanics (see [11] and [12]).

Motivated by above mentioned result, in this paper, we study the asymptotic behavior (global attractor) for a class of parabolic equations involving (sub-)critical Sobolev and Hardy exponent:

$$\begin{cases} u_t - \Delta_p u + f(u) = V(x)\dfrac{|u|^{q-2}}{|x|^s}u, & x \in \mathbb{R}^N, t > 0, \\ u(0,x) = u_0 & x \in \mathbb{R}^N \end{cases} \tag{1.5}$$

where $1 < p < N(N \geq 2)$. Comparing to (1.1) and (1.2), we consider the Cauchy problem in the whole space $\mathbb{R}^N$ while we weaken the conditions on the relationship of p, q and s as follows (see (1.3))

$$2 \leq q \leq p_*(s) := \frac{p(N-s)}{N-p} \ \text{ with } \ 0 < s < \min\{p, q\}, \tag{1.6}$$