Nonlinear Analysis: Real World Applications 42 (2018) 485–510

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

Transport of congestion in two-phase compressible/incompressible flows

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ARTICLE INFO

Article history:
Received 3 January 2017
Received in revised form 22 January 2018
Accepted 5 February 2018

Keywords:
Constrained fluid model
Navier–Stokes equations
Free boundary
Singular pressure
Renormalized transport

ABSTRACT

We study the existence of weak solutions to the two-phase fluid model with congestion constraint. The model encompasses the flow in the uncongested regime (compressible) and the congested one (incompressible) with the free boundary separating the two phases. The congested regime appears when the density in the uncongested regime $\varrho(t,x)$ achieves a threshold value $\varrho^*(t,x)$ that describes the comfort zone of individuals. This quantity is prescribed initially and transported along with the flow. We prove that this system can be approximated by the fully compressible Navier–Stokes system with a singular pressure, supplemented with transport equation for the congestion density. We also present the application of this approximation for the purposes of numerical simulations in the one-dimensional domain.

1. Introduction

Our aim is to analyse the free-boundary two-phase fluid system that could be used to model the congestions in the large group of individuals in a bounded area. Individuals are just the agents that have their own preferences for how close they let the closest neighbour to approach and they carry this information with them in the course of motion. They do not follow any neighbour trying to align their velocities, nor they are trying to reach a certain target, as for example, the evacuation point. We simply prescribe their initial velocity that determines their direction of motion and check how the individual preferences as well as the initial distribution of the agents determines creation of congestions. Such model could be used as a building block of more involved crowds modelling [1], some progress in this direction has been made in our recent numerical work [2].

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https://doi.org/10.1016/j.nonrwa.2018.02.001

Crowd modelling is a problem of strategic importance for safety reasons. It has been studied in many parallel approaches. We can distinguish, for example, the mean-field game models [3,4], in which the individuals behave as the players following some strategy, or optimizing certain cost; the microscopic models which describe precise position and velocity of an individual (Individual-Based-Models) using Newtonian framework [5–8]; or the macroscopic models formulated in the language developed for the fluids [9–13]. The behaviour of the crowd in the later is characterized by some averaged quantities such as the number density or mean velocity. The macroscopic models, although less precise than the microscopic ones, are computationally more affordable. Moreover, they allow for asymptotic studies that proved to be useful for understanding various aspects like: swarming or pattern formation observed in the experiments.

It is an extensive field of research to develop continuum models that are able to exhibit features of the kinetic approach. Although it would be desirable to use computationally cheap fluid approach to describe the crowd dynamics, the nowadays models are not developed enough to recreate behaviour observed in real world. In this paper we present, as far as we know, the first mathematical result for the fluid model that incorporates various sizes of the individuals/particles and their inhomogeneities. Similar approach has been recently applied in [14] in the context of granular media flow with memory effects. In order to use this model for more specific applications, its current version needs to be supplemented with agents specific features and we postpone this topic to further research.

Our system writes as follows:

$$\partial_t \varrho + \mathrm{div}(\varrho \mathbf{u}) = 0, \tag{1a}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla \pi + \nabla p\left(\frac{\varrho}{\varrho^*}\right) - \mathrm{div}\mathbf{S}(\mathbf{u}) = \mathbf{0}, \tag{1b}$$

$$\partial_t \varrho^* + \mathbf{u} \cdot \nabla \varrho^* = 0, \tag{1c}$$

$$0 \le \varrho \le \varrho^*, \tag{1d}$$

$$\mathrm{div}\mathbf{u} = 0 \text{ in } \{\varrho = \varrho^*\}, \tag{1e}$$

$$\pi \ge 0 \text{ in } \{\varrho = \varrho^*\}, \quad \pi = 0 \text{ in } \{\varrho < \varrho^*\}. \tag{1f}$$

with the unknowns: $\varrho = \varrho(t,x)$ – the mass density, $\mathbf{u} = \mathbf{u}(t,x)$ – the velocity vector field, $\varrho^* = \varrho^*(t,x)$ – the congestion density, also referred to as the barrier or the threshold density, and π – the congestion pressure, that appears only when $\varrho = \varrho^*$.

The barotropic pressure is an explicit function of $\frac{\varrho}{\varrho^*}$

$$p\left(\frac{\varrho}{\varrho^*}\right) = \left(\frac{\varrho}{\varrho^*}\right)^\gamma, \quad \gamma > 1, \tag{2}$$

and plays the role of the background pressure.

The stress tensor $\mathbf{S}$ is a known function of $\mathbf{u}$, characteristic for the Newtonian fluid, namely

$$\mathbf{S} = \mathbf{S}(\mathbf{u}) = 2\mu\,\mathrm{D}(\mathbf{u}) + \lambda \mathrm{div}\mathbf{u}\,\mathrm{I}, \quad \mu > 0,\ 2\mu + \lambda > 0, \tag{3}$$

where $\mathrm{D}(\mathbf{u}) = (\nabla \mathbf{u} + \nabla^T \mathbf{u})/2$ denotes the symmetric part of the gradient of $\mathbf{u}$, and $\mathrm{I} = \mathrm{I}_3$ is the identity matrix.

In the system (1) variable ϱ^* models preferences of the individuals, it is given initially and then transported with the flow. Therefore, ϱ^* depends on time and position, but more importantly it depends on initial configuration ϱ_0^*. The form of ϱ^* relaxes the restrictions from the models studied in [15,16], where the threshold density ϱ^* was either assumed to be constant or independent of time. This allows to cover more