Nonlinear Analysis: Real World Applications 41 (2018) 1–33

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

Unconditional small data global well-posedness for nonlinear beam equations with weak damping

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ARTICLE INFO

Article history:
Received 5 June 2017
Received in revised form 8 October 2017
Accepted 13 October 2017

Keywords:
Unconditional global well-posed
Cauchy problem
Fourth order wave equation with damping
Large time behavior

ABSTRACT

In this paper, we investigate the large time behavior of a global solution for the nonlinear beam equation with a weak damping term in weighted Sobolev spaces. We proved the unconditional global well-posedness for small initial data and sharp decay estimates of the global solution in the same framework. The crucial part of our proof is a smoothing effect from the linear principal part to overcome the regularity loss structure in the weighted estimates.

1. Introduction

In this paper we study the Cauchy problem for the following semilinear damped beam equation

$$\partial_t^2 u + \partial_t u + \partial_x^4 u - \alpha \partial_x^2 u = \partial_x f(\partial_x u), \qquad t > 0, \quad x \in \mathbb{R}, \tag{1.1}$$

$$u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), \qquad x \in \mathbb{R}, \tag{1.2}$$

where $u = u(t,x) : (0,T) \times \mathbb{R} \to \mathbb{R}$ is the unknown function, $u_0(x)$ and $u_1(x)$ are given initial data and α is a positive constant. Without loss of generality, we assume $\alpha = 1$ for simplicity. We also assume that the nonlinear function $f \in C^\infty(\mathbb{R})$ satisfies that for some integer $p \geq 2$,

$$f(v) = O(|v|^p) \qquad \text{as } v \to 0. \tag{1.3}$$

Eq. (1.1) arises from the movement of the study of the mechanical movements of shape memory alloys of a constant mass density (cf. [1]). For the details of the physical background of Eq. (1.1), see [1,2] and the reference therein. In the mathematical point of view, we can expect that the solutions of Eq. (1.1) have the

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https://doi.org/10.1016/j.nonrwa.2017.10.007
1468-1218/

dissipative properties which arise from the weak damping term $+\partial_t u$ and, at the same time, the dispersive properties from the 4th order derivative term $+\partial_x^4 u$.

Our purpose here is to show sharp decay estimates of the global solutions, which satisfy the unconditional well-posedness, using the decay structure and a smoothing effect from the linear principal parts. Here we will follow the notion of well-posedness given in [3] (cf. [4–7]): let X and $\tilde{X}$ be function spaces. The initial value problem is said to locally well posed in X if for each $(u_0, u_1) \in X \times \tilde{X}$ there exist $T > 0$ and a unique solution $u \in C([0,T); X) \cap \cdots = Y_T$ of the equation, with the map data-solution $(u_0, u_1) \to u$, being continuous. In the case, where we can take $T > 0$ arbitrarily large, we say the problem is globally well posed. Especially we call the problem satisfies the unconditional global well-posedness if $C([0,\infty); X) = Y_\infty$ holds.

Before formulating the results of this paper, let us review several known results on (1.1). In the bounded interval with the 0-Neumann boundary condition, Racke and Shang [1] proved the global well-posedness for large data in $H^4 \times H^2$ and the existence of global attractors, applying the energy method. After that, the author of this paper and Yoshikawa studied the Cauchy problem (1.1)–(1.2) in the series of the papers [2,8,9]. If the nonlinearity is given by $f(v) = |v|^{p-1}v$ with $p \geq 2$, they proved the unconditional global well-posedness in the energy class $H^2 \times L^2$ for large data and energy decay estimates in [9] using the energy method due to Kawashima–Nakao–Ono [10] (see also [11]). On the other hand, for the nonlinearity including the minus sign case e.g. $f(v) = -|v|^{p-1}v$ with $p \geq 2$, we cannot expect that the classical energy method works well, since the standard energy is not necessarily non-negative definite. In this situation, Takeda–Yoshikawa ([2,8]) constructed the unique global solution in the class $C([0,\infty); L^1 \cap H^2)$ for small initial data $(u_0, u_1) \in W^{2,1} \cap H^2 \times L^1 \cap L^2$ and they also obtained optimal decay estimates and a smoothing effect of the global solution. Especially, they proved the following two facts. First, when t is large, the solution behaves likes the Gauss kernel

$$G_t(x) := \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x|^2}{4t}} \tag{1.4}$$

with the multiple of the constant

$$M_0 := \int_{\mathbb{R}} u_0(x) + u_1(x)\, dx. \tag{1.5}$$

This means that $u(t) - M_0 G_t$ decays faster than $u(t)$ and $M_0 G_t$ as $t \to \infty$.

Second, under the additional assumption for the initial data $u_0, u_1 \in L^1_1 := \{(1+|x|)u \in L^1\}$, the solution satisfies the second order expansion by the Gauss kernel. In other words, $u(t) - \sum_{j=0}^{1} M_j \partial_x^j G_t - \tilde{M}\partial_x G_t$ decays faster than $u(t) - M_0 G_t$ and $(M_1 + \tilde{M})\partial_x G_t$ as $t \to \infty$, where

$$M_1 := \int_{\mathbb{R}} (-x)(u_0(x) + u_1(x))\, dx, \tag{1.6}$$

$$\tilde{M} := \int_0^\infty \int_{\mathbb{R}} f(\partial_x u(\tau, x)) dx d\tau. \tag{1.7}$$

Later on, Zhang–Li [12] proved the unique existence of the global solution in the class $C([0,\infty); H^2)$ for small initial data $(u_0, u_1) \in L^1 \cap H^2 \times L^1 \cap L^2$ and obtained its asymptotic profile. The author of this paper also obtained the global in time solution and its large time behavior for the slowly decaying data, which means that initial data $u_0(x), u_1(x)$ behaves like $(1 + x^2)^{-k}$ for $0 < k \leq 1$ in [13].

Here we note that for the Cauchy problem (1.1)–(1.2), the authors in the previous results focused on the diffusive structure of the solution and they did not discuss the well-posedness. It is well-known that, to obtain sharp decay properties in [2,8,9,12], the L^1 regularity assumption for initial data is essential. On the other hand, it seems difficult to show the global well-posedness in the L^1 framework, since the equation has the dispersive structure, not only the dissipative one. Then we use the weighed Sobolev space $H^{s,\ell}$ defined by

$$H^{s,\ell} := \{u \in L^2(\mathbb{R}) | \|(1 + x^2)^{\frac{\ell}{2}} \partial_x^k u\|_2 < \infty \ \text{ for } 0 \leq k \leq s\}$$