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Global regularity of 2D almost resistive MHD equations

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ABSTRACT

Whether or not the solution to 2D resistive MHD equations is globally smooth remains open. This paper establishes the global regularity of solutions to the 2D almost resistive MHD equations, which require the dissipative operators $\mathcal{L}$ weaker than any power of the fractional Laplacian by taking advantage of nonlinear maximum principles. The result is an improvement of the one of Fan et al. (2014) which ask for $\alpha > 0, \beta = 1$.

1. Introduction

Consider the Cauchy problem of the two-dimensional generalized magnetohydrodynamic equations:

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + b \cdot \nabla b - \nu \Lambda^{2\alpha} u, \\ b_t + u \cdot \nabla b = b \cdot \nabla u - \kappa \Lambda^{2\beta} b, \\ \nabla \cdot u = \nabla \cdot b = 0, \\ u\left(x,0\right) = u_0\left(x\right), \quad b\left(x,0\right) = b_0\left(x\right) \end{cases} \tag{1.1}$$

for $x \in \mathbb{R}^2$ and $t > 0$, where $u = u(x,t)$ is the velocity, $b = b(x,t)$ the magnetic, $p = p(x,t)$ the pressure, and $u_0(x), b_0(x)$ with $\mathrm{div} u_0(x) = \mathrm{div} b_0(x) = 0$ are the initial velocity and magnetic, respectively. Here $\nu, \kappa, \alpha, \beta \geq 0$ are nonnegative constants and $\Lambda = \sqrt{-\Delta}$.

The global regularity of the d-D GMHD (1.1) has attracted a lot of attention and progress has been made in the last few years (see [1–11]). In 2D case, it follows from [1,2,5] that the problem (1.1) has a unique global regular solution if $\alpha = 0,\ \beta > 1$ or $\alpha > 0,\ \beta = 1$. Recently, some important progresses have been made on the global well-posedness for non-resistive MHD equations ($\kappa = 0,\ \alpha = 1$) near an equilibrium (see [12–14]). Some results on global regularity of 2D MHD equations with partial viscosity and resistivity refer to [15,16]. To the best of our knowledge, whether or not there exists a global regular solution for 2D resistive MHD ($\nu = 0,\ \beta = 1$) is still an open problem.

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In this paper, we are concerned with the following 2D GMHD

$$\begin{cases} u_t + u \cdot \nabla u + \mathcal{L}u = -\nabla p + b \cdot \nabla b, \\ b_t + u \cdot \nabla b - \Delta b = b \cdot \nabla u, \\ \nabla \cdot u = \nabla \cdot b = 0, \\ u(x,0) = u_0(x), \;\; b(x,0) = b_0(x), \end{cases} \tag{1.2}$$

where $\mathcal{L}$ is the dissipative operator with

$$\mathcal{L}u(x) = P.V. \int_{\mathbb{R}^2} \frac{u(x) - u(x-y)}{|y|^2 m(|y|)} \mathrm{d}y. \tag{1.3}$$

Here $m : [0,\infty) \to [0,\infty)$ is a smooth, non-decreasing function that behaves like $\frac{1}{(-\log r)^{1+\varepsilon_1}}$ for sufficiently small r with $\varepsilon_1 > 0$ and that grows fast at least at the rate of $(\log r)^{1+\varepsilon_2}$ for sufficiently large r with $\varepsilon_2 > 0$, satisfying

$$\int_0^1 \frac{m(r)}{r} \mathrm{d}r < \infty \tag{1.4}$$

and the doubling condition

$$m(2r) < cm(r) \tag{1.5}$$

for some positive constants c.

The main result of this paper is stated as follows.

Theorem 1.1. *Let $m(r)$ satisfy (1.3)–(1.5) and $\rho \geqslant 4$. Assume that $u_0, b_0 \in H^\rho(\mathbb{R}^2)$ with $\mathrm{div} u_0 = \mathrm{div} b_0 = 0$. Then for any $T > 0$, the Cauchy problem (1.2) has a unique regular solution*

$$(u,b) \in C([0,T]; H^\rho(\mathbb{R}^2)) \text{ and } b \in L^2([0,T]; H^{\rho+2}(\mathbb{R}^2)).$$

The existence and uniqueness are standard, so we omit their proofs, and only give the a priori estimates.

Remark 1.1. Due to

$$\Lambda^{2\alpha} u(x) = c_\alpha P.V. \int_{\mathbb{R}^2} \frac{u(x) - u(x-y)}{|y|^{2+2\alpha}} \mathrm{d}y \tag{1.6}$$

for $\alpha \in (0,1)$ (see [17]), the dissipative operator $\mathcal{L}$ defined in Theorem 1.1 is weaker than any power of the fractional Laplacian. Thus we improve the results in [2] for Eqs. (1.1) which require $\alpha > 0, \beta = 1$.

Remark 1.2. Inspired by the work [18,19], (1.4) can be replaced by weaker conditions $\lim_{r\to 0^+} m(r) = 0$, therefore we can obtain the global regularity of solutions to (1.2) with arbitrary weak dissipation $\mathcal{L}$ (see Remark 3.1).

Remark 1.3. In virtue of Remark 2.1 and Section 3, we require only $u_0, b_0 \in H^\rho(\mathbb{R}^2)$ with $\rho > 3$.

Remark 1.4. For the 2D GMHD (1.2), it remains an open problem whether there exists a global smooth solution without the dissipative operator $\mathcal{L}$.

The idea to prove Theorem 1.1 is motivated by [18]. Recently, Constantin and Vicol [18] provided the transparent proofs of global regularity for critical SQG and critical d-dimensional Burgers equations by taking advantage of nonlinear maximum principles, which also can be used to obtain the global regularity