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Global existence for a singular phase field system related to a sliding mode control problem☆

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ABSTRACT

In the present contribution we consider a singular phase field system located in a smooth and bounded three-dimensional domain. The entropy balance equation is perturbed by a logarithmic nonlinearity and by the presence of an additional term involving a possibly nonlocal maximal monotone operator and arising from a class of sliding mode control problems. The second equation of the system accounts for the phase dynamics, and it is deduced from a balance law for the microscopic forces that are responsible for the phase transition process. The resulting system is highly nonlinear; the main difficulties lie in the contemporary presence of two nonlinearities, one of which under time derivative, in the entropy balance equation. Consequently, we are able to prove only the existence of solutions. To this aim, we will introduce a backward finite differences scheme and argue on this by proving uniform estimates and passing to the limit on the time step.

1. Introduction

This paper is devoted to the mathematical analysis of a system of partial differential equations (PDE) arising from a thermodynamic model describing phase transitions. The system is written in terms of a rescaled balance of energy and of a balance law for the microforces that govern the phase transition. Moreover, the first equation of the system is perturbed by the presence of an additional maximal monotone nonlinearity. This paper will focus only on analytical aspects and, in particular, will investigate the existence of solutions. In order to make the presentation clear from the beginning, we briefly introduce the main ingredients of the PDE system and give some comments on the physical meaning.

We deal with a two-phase system located in a smooth bounded domain $\Omega \subseteq \mathbb{R}^3$ and let $T > 0$ denote some final time. The unknowns of the problem are the *absolute temperature* ϑ and an *order parameter* χ

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which can represent the local proportion of one of the two phases. To ensure thermomechanical consistency, suitable physical constraints on χ are considered: if it is assumed, e.g., that the two phases may coexist at each point with different proportions, it turns out to be reasonable to require that χ lies between 0 and 1, with $1-\chi$ representing the proportion of the second phase. In particular, the values $\chi = 0$ and $\chi = 1$ may correspond to the pure phases, while χ is between 0 and 1 in the regions when both phases are present. Clearly, the system provides an evolution for χ that has to comply with the previous physical constraint.

Now, let us state precisely the equations as well as the initial and boundary conditions. The equations governing the evolution of ϑ and χ are recovered as balance laws. The first equation comes from a reduction of the energy balance equation divided by the absolute temperature ϑ (see [1, formulas (2.33)–(2.35)]). Therefore, the so-called entropy balance can be written in $\Omega \times (0,T)$ as follows:

$$\partial_t(\ln\vartheta + \ell\chi) - k_0\Delta\vartheta = F, \tag{1.1}$$

where ℓ is a positive parameter, $k_0 > 0$ is a thermal coefficient for the entropy flux $\mathbf{Q}$, which is related to the heat flux vector $\mathbf{q}$ by $\mathbf{Q} = \mathbf{q}/\vartheta$, and F stands for an external entropy source.

In the present contribution, we assume that the entropy balance equation (1.1) is perturbed by the presence of an additional maximal monotone nonlinearity, i.e.,

$$\partial_t(\ln\vartheta + \ell\chi) - k_0\Delta\vartheta + \zeta = F, \tag{1.2}$$

where

$$\zeta(t) \in A(\vartheta(t) - \vartheta^*) \qquad \text{for a.e. } t \in (0,T). \tag{1.3}$$

Here, ϑ^* is a positive and smooth function ($\vartheta^* \in H^2(\Omega)$ with null outward normal derivative on the boundary) and $A : L^2(\Omega) \to L^2(\Omega)$ is a maximal monotone operator satisfying some conditions, namely: A is the subdifferential of a proper, convex and lower semicontinuous (l.s.c.) function $\Phi : L^2(\Omega) \to \mathbb{R}$ which takes its minimum in 0, and A is linearly bounded in $L^2(\Omega)$. In order to explain the role of this further nonlinearity, we refer to [2], where a class of sliding mode control problems is considered: a state-feedback control $(\vartheta,\chi) \mapsto u(\vartheta,\chi)$ is added in the balance equations with the purpose of forcing the trajectories of the system to reach the sliding surface (i.e., a manifold of lower dimension where the control goal is fulfilled and such that the original system restricted to this manifold has a desired behavior) in finite time and maintains them on it. As widely described in [2], this study is physically meaningful in the framework of phase transition processes.

Let us mention the contributions [3,4], where standard phase field systems of Caginalp type, perturbed by the presence of nonlinearities similar to (1.3), are considered. In [3,4] the existence of strong solutions, the global well-posedness of the system and the sliding mode property can be proved; unfortunately, here the problem we consider is rather more delicate due to the doubly nonlinear character of Eq. (1.2) and it turns out that we cannot perform a so complete analysis. On the other hand, we observe that, due to the presence of the logarithm of the temperature in the entropy equation (1.2), in the system we investigate here the positivity of the variable representing the absolute temperature follows directly from solving the problem, i.e., from finding a solution component ϑ to which the logarithm applies. This is an important feature and avoids the use of other methods or the setting of special assumptions, in order to guarantee the positivity of ϑ in the space-time domain.

The second equation of the system under study describes the phase dynamics and is deduced from a balance law for the microscopic forces that are responsible for the phase transition process. According to [5,6], this balance reads

$$\partial_t\chi - \Delta\chi + \beta(\chi) + \pi(\chi) \ni \ell\vartheta, \tag{1.4}$$