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Travelling wave solutions for the Richards equation incorporating non-equilibrium effects in the capillarity pressure

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A B S T R A C T

The Richards equation is a mathematical model for unsaturated flow through porous media. This paper considers an extension of the Richards equation, where non-equilibrium effects like hysteresis and dynamic capillarity are incorporated in the relationship that relates the water pressure and the saturation. The focus is on travelling wave solutions, for which the existence is investigated first for the model including hysteresis and subsequently for the model including dynamic capillarity effects. In particular, such solutions may have non monotonic profiles, which are ruled out when considering standard, equilibrium type models, but have been observed experimentally. The paper ends with numerical experiments confirming the theoretical results.

1. Introduction

Unsaturated flow through porous media is encountered in many applications of societal and engineering relevance. Examples in this sense are the groundwater flows, or the moisture dynamics in building materials. A commonly used mathematical model for such kind of processes is the Richards equation, which is obtained after inserting the Darcy law into the water mass balance equation. The two main unknowns in this equation are the water saturation S (the fraction of the pore space in a representative elementary volume that is occupied by water) and the water pressure p. In standard porous media flow models, these two unknowns are related through the strictly decreasing capillary pressure function $P_c(\cdot)$, namely $p = -P_c(S)$, which is determined experimentally. Different types of functions and parameterisations are discussed e.g. in [1], the common assumption being that the dependence is obtained under special, equilibrium conditions. More precisely, the experiments are carried out either for imbibition or for drainage and not when these processes

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occur alternatively, and during an entire imbibition or drainage cycle each measurement has been done only after water stops redistributing inside the pores of the elementary volumes. Such models will therefore be called in what follows "equilibrium type models".

In realistic applications, neither of these conditions are met. First, experiments reported e.g. in [2,3] have revealed the hysteretic nature of the pressure–saturation relationship. More precisely, it was observed that the functions P_c determined during infiltration and drainage are different. This motivated an extremely rich literature on mathematical models describing hysteresis. The play-type hysteresis model assumes a switch between imbibition and drainage capillary pressure–saturation curves whenever the saturation changes from increasing in time to decreasing in time or vice-versa. A mathematical formulation of this is given in [4], and the switch happens along vertical scanning curves. This poses nontrivial issues when analysing the resulting models and their numerical discretisations, which can be resolved by approximating the vertical scanning curves by monotone and non-vertical ones. In this sense, commonly used is the Lenhard–Parker model [5], where the scanning curves are rescaled versions of some predefined curves. A simplified version of it is proposed in [6], where the scanning curves are oblique lines. Also extension of play-type hysteresis model incorporating the non-vertical scanning curves has been proposed in [7]. Other hysteresis models build on concepts like percolating/nonpercolating phases [8,9], or interfacial area based models [10,11]. An overview of hysteresis models can be found in [12], whereas details on the numerical approximation of hysteresis in porous media models are given in [13]. In the present paper we consider the play-type hysteresis model for the pressure–saturation dependence but it is interesting to note that hysteresis can also be present in the relative permeability curve [14]. However, in the latter case this effect is less important in comparison to the former [2].

Second, when letting the water infiltrate in a homogeneous medium, experiments have revealed profiles that are conflicting with the profiles of the solutions to the equilibrium type models. For example, if the injection rates at the inflow are high enough, the obtained saturation profiles are non-monotone as the values at some locations inside the column are higher than at the inflow boundary (the so-called overshoot phenomenon, see [15]). In particular, the experiments in [16] show that although the saturation at some certain location is decreasing in time, the water pressure is non-monotone and exhibits a peak at moments when the saturation changes rapidly. This pleads for the inclusion of dynamic effects in the pressure–saturation relationship, as suggested in [17].

In mathematical terms, models like those mentioned above are evolution equations of pseudo-parabolic type, or involve differential inclusions. Such models will be called below "non-equilibrium type models". In this paper, we investigate how the solution profiles for unsaturated flow through a long, homogeneous porous column are affected by such non-equilibrium effects. The analysis is based on travelling waves (TW), allowing to reduce the model first to a nonlinear ordinary differential equation, and then to a dynamical system. This provides insight in the structure and behaviour of the solutions, and in particular how the non-equilibrium regime affects the profiles. The present analysis follows the ideas in [18], which studies the existence of TW solutions for reactive flow and transport models in porous media. In [19] TW solutions are analysed for nonlinear models that are similar to the Richards equation, but where higher order effects are included inspired by the ones describing dynamic capillarity. The nonlinear functions taken in [19] are of power-like type, in particular the flux function is convex. The existence of TW solutions is analysed, and in particular it is shown that oscillations behind the infiltration front may occur, depending on the magnitude of the dynamic effect. A similar analysis, but for two-phase flow models implying convex–concave flux functions is carried out in [20–22]. Also related are the diffusive–dispersive equations appearing as models for the phase transition dynamics, but in which the higher order terms are in terms of the spatial derivatives only [23,24]. Though having a different motivation, the associated TW equation is similar to the one for the dynamic capillarity models, in particular since both involve a non-convex nonlinearity in the lower order terms. In this context, in [22] it is proved that the saturation profile may have overshoot in form of a plateau