Nonlinear Analysis: Real World Applications 41 (2018) 412–427

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

On analytical solutions of a two-phase mass flow model[☆]

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ARTICLE INFO

Article history:
Received 5 March 2017
Accepted 8 September 2017

Keywords:
Analytical solutions
Debris flow
Lie symmetry
Two-phase mass

ABSTRACT

We investigate a two-phase mass flow model by constructing analytical solutions with their physical significance. We use the method of splitting and separation of variables to reduce the system of non-linear PDEs modeling the two-phase mass flow into quasilinear PDEs. In particular, the system of non-linear PDEs is reduced into Riccati equations and Burgers equations, thereby making it possible to solve. Starting with simple analytical solutions, we construct analytical solutions with increased complexities for the phase velocities and the phase heights as functions of space and time. Furthermore, we use the Lie group action to generate more analytical solutions and analyze their possible invariance. We also present a perspective called relative non-invariance associated to the underlying physics relevant to multi-phase flows, namely, the relative velocity and relative flow depths between the phases. Finally, we present detailed analysis and discussion on the time and spatial evolutions of the analytical solutions for solid and fluid phase velocities and the flow depths. The obtained analytical solutions corroborate with the physics of two-phase mass flows down a slope.

1. Introduction

Debris flows are two-phase, gravity-driven mass flows consisting of solid particles and fluid. Debris flows play an important role in environment, geophysics and engineering [1–11]. Two-phase flows are generally characterized by the relative motion between the solid and fluid phases, which depends on the mixture composition, solid–fluid interactions, and the dynamics as modeled by the main driving forces. These flows are extremely destructive and dangerous in nature, hence, predicting the nature of the flow, its dynamics, and runout distances could be very valuable. The solid and fluid phase velocities in debris flows may deviate substantially from each other essentially affecting the flow mechanics. This makes debris flows a challenging research area. There has been extensive field, experimental, theoretical, and numerical investigations on the dynamics, consequences, and industrial applications of such mixture flows [11–17].

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https://doi.org/10.1016/j.nonrwa.2017.09.009
1468-1218/

Various research in the past few decades focused on different aspects of two-phase debris flows [1–9]. More recently, in Pudasaini (2012) [18], two-phase mass flows have been studied with a comprehensive model that explicitly includes different interactions between the solid particles and the viscous fluid. The model in [18] constitutes the most generalized two-phase flow model to date. This model reproduces previous simple models, which was considered for single- and two-phase avalanches and debris flows, as special cases. Recently, extensive simulations and applications of the general two-phase mass flows have been carried out for different mass flows including landslide impacting a reservoir [19], glacial lake outburst floods [20], and complex and multiple process chains with breaching of dam resulting in subsequent debris floods [17]. Similarly, several analytical solutions have been constructed for two-phase or reduced mass flows [21–24]. Exact and analytical solutions of the complex two-phase mass flow models are very important as these solutions highlight the insight of the underlying physical model. Furthermore, these exact solutions can be utilized to calibrate and validate the numerical schemes and numerical simulations before such simulations methods can be applied to real world problems. For these reasons, here, we are concerned in constructing new analytical solutions for the simplified form of the two-phase mass flow model [18].

In this paper, we advance the investigation of the two-phase mass flow model from [18,23] by constructing several families of new explicit solutions and studying their physical significance. We apply the method of splitting and method of separation of variables to solve the system into consideration. The method of splitting allows us to reduce the two-phase mass flow model into a decoupled system of first order quasilinear PDEs. Further, we reduce these quasilinear PDEs into Burgers equations and Riccati equations. This allows us to construct solutions of the two-phase mass flow model. Analytical solutions are obtained with increased complexities for the phase velocities and the phase heights in two-phase mixture mass flows as functions of space and time. Furthermore, these solutions for flow velocities and depth evolutions are shown to be in line with the physics of two-phase mass flows down a channel.

The Lie symmetry method is a widely used tool to study differential equations arising from wide range of physical problems. Its application includes the reduction of differential equations, order reduction of ordinary differential equations, construction of invariant solutions, and mapping between the solutions in mechanics, applied mathematics and mathematical physics, and applied and theoretical physics [10,25–32]. In [23], Ghosh Hajra et al. initiated studying the model in [18] using the Lie symmetry method where the most general symmetry Lie algebra of the system is computed. Moreover, several physically significant analytic and numerical solutions are presented. More recently, in [24], the optimal systems of the symmetry Lie algebra has been used to reduce the model (from [23]) into other systems of PDEs and ODEs. Also, these optimal Lie subalgebras are used to generate several physically relevant numerical solutions. In this paper, we use the Lie symmetry algebra constructed in [23] to generate Lie symmetry transformed solutions [25,31] from the newly obtained solutions. From the viewpoint of the number of parameter describing solutions, some transformed solutions are invariant while others are not. Furthermore, based on the structure of transformed solutions and the difference between the pair of solid and fluid phase velocities, we introduce a notion called relative non-invariance mapping.

2. Two-phase mass flow model

In this section, we briefly review the basic features of a two-phase mass flow model into consideration, as in Ghosh Hajra et al. [23,24]. This two-phase mass flow model is the one-dimensional inclined channel flow model of [11], which is a special case of the general model in [18]. Let t be time, X and Z be coordinates along and normal to the slope with inclination ζ. In the following, the solid and fluid constituents are denoted by the suffices s and f, respectively. Let h be the mixture flow depth, $h_s = \alpha_s h, h_f = \alpha_f h$ be the solid and fluid flow depths, and $Q_s = h_s u_s = \alpha_s h u_s, Q_f = h_f u_f = \alpha_f h u_f$ are the corresponding fluxes, where α_s, $\alpha_f (= 1 - \alpha_s)$ are the solid and fluid volume fractions, respectively. The solid and fluid net driving forces are