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Global classical solution to a chemotaxis consumption model with singular sensitivity

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ABSTRACT

In this paper, we are concerned with the chemotaxis consumption model with singular sensitivity

$$\begin{cases} u_t = \Delta u - \nabla \cdot (\frac{f(u)}{v}\nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - uv, & x \in \Omega,\ t > 0, \end{cases}$$

under homogeneous Neumann boundary conditions in a smooth bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$, for $0 < f(u) \leq K(u+1)^\alpha$ with some $K > 0$ and $\alpha < 2$. It is shown that for any sufficiently smooth initial data, the above system admits a global classical solution when either $n = 1$ and $\alpha < 2$, or $n \geq 2$ and $\alpha < 1 - \frac{n}{4}$.

1. Introduction

Chemotaxis, known as a biassed movement of cells toward a concentration gradient of chemical cue, plays a crucial role in numerous biological circumstances. A well-known chemotaxis model was initially proposed by Keller and Segel in 1970 [1] and the blow-up of solutions of this model in finite time is a striking indication of the spontaneous formation of cell aggregates [2–5]. Lately, various biological mechanisms including nonlinear diffusion [6,7], volume-filling [8], logistic dampening [9,10] and saturating signal production [11,12] are expected to prevent such chemotactic collapse.

In the present work, we consider the following chemotaxis consumption system with singular sensitivity

$$\begin{cases} u_t = \Delta u - \nabla \cdot (\frac{f(u)}{v}\nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - uv, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

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in a smooth bounded domain $\Omega \subset \mathbb{R}^n, n \geq 1$, $u = u(x,t)$ denotes the cell density, $v = v(x,t)$ represents the signal concentration. Throughout this paper we assume that f satisfies

$$f \in C^{1+\theta}([0,+\infty)), \quad f(0) = 0 \quad \text{and} \quad 0 < f(s) \leq K(s+1)^{\alpha} \quad \text{for any } s > 0, \tag{1.2}$$

where $\theta \in (0,1)$, $K > 0$ and $\alpha < 2$ are constants. Moreover, the initial data u_0 and v_0 satisfy

$$\begin{cases} u_0 \in C^0(\overline{\Omega}), & u_0 \geq 0 \quad \text{in } \Omega \quad \text{and} \quad u_0 \not\equiv 0 \quad \text{as well as} \\ v_0 \in W^{1,\infty}(\Omega), & v_0 > 0 \quad \text{in } \overline{\Omega}. \end{cases} \tag{1.3}$$

When $f(u) = u$, Winkler proved that the system (1.1) has at least one global generalized solution for $n = 2$, and moreover v converges to 0 with respect to the norm in $L^p(\Omega), p \in [1,\infty)$, and to the weak-$*$ topology of $L^{\infty}(\Omega)$ [13]. Furthermore, it is shown that the solution is smooth and converges to the homogeneous steady state whenever the initial data is appropriately small [14]. From the angle of classical solution, Lankeit showed the existence of locally bounded global solutions of the system

$$\begin{cases} u_t = \nabla \cdot (H(u)\nabla u) - \nabla \cdot (\frac{u}{v}\nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - uv, & x \in \Omega,\ t > 0, \end{cases} \tag{1.4}$$

where $H(u) \geq \delta u^{m-1}, \delta > 0, n \geq 2$, under the condition $m > 1 + \frac{n}{4}$ [15]. Accordingly, a natural question is whether a corresponding conclusion holds in the system (1.1). The purpose of this work is to answer the issue of global classical solutions for $n \geq 1$. The details read as follows.

Theorem 1.1. *Suppose that $\Omega \subset \mathbb{R}^n, n \geq 1$, is a bounded domain with smooth boundary. Let f satisfy (1.2) with some $\theta \in (0,1)$, $K > 0$ and*

$$\begin{cases} \alpha < 2, & \text{if} \quad n = 1, \\ \alpha < 1 - \dfrac{n}{4}, & \text{if} \quad n \geq 2. \end{cases} \tag{1.5}$$

Then for any (u_0, v_0) fulfilling (1.3), there exists a couple $(u,v) \in (C^0(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty)))^2$ which solves (1.1) classically. Moreover, we have $u \geq 0$ and $v > 0$ in $\bar{\Omega} \times [0,\infty)$.

It is presented in [16] and [7] that under homogeneous Neumann boundary conditions the quasilinear system

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \end{cases}$$

with $D(u) = (u+\epsilon)^{-p}, S(u) = (u+\epsilon)^q, u \geq 0, \epsilon > 0, p \in \mathbb{R}, q \in \mathbb{R}$ and $n \geq 1$, possesses a globally bounded classical solution provided that $p + q < \frac{2}{n}$. Meanwhile, as to the chemotaxis consumption models (1.1) and (1.4), it is shown that the results in Theorem 1.1 and [15] reach the same criticality, when $n \geq 2$, i.e. $\frac{f(u)}{u^0}$ and $\frac{u}{H(u)}$ can be controlled by cu^r with some $c > 0$ and $r < 1 - \frac{n}{4}$.

We underline that the above result in Theorem 1.1 heavily relies on the assumption (1.5). However, whether or not the exponents $\alpha = 2$ (when the spatial dimension $n = 1$) and $\alpha = 1 - \frac{n}{4}$ (when the spatial dimension $n \geq 2$) are indeed critical for the global classical solution of the system (1.1) remains open.

Before closing this introductory section, let us briefly mention the main ideas of the proof for our result. We divided it into two steps. Firstly, let $w(x,t) := -\ln\Big(\frac{v(x,t)}{\|v_0\|_{L^{\infty}(\Omega)}}\Big)$ and then from the system (1.1) we have (2.2) (see below) with the nonsingular chemotaxis term $\nabla \cdot (f(u)\nabla w)$ instead of $\nabla \cdot (\frac{f(u)}{v}\nabla v)$, which seems more accessible. We next use this nonsingular system (2.2) to obtain the local boundedness of v, motivated by [15]. Secondly, we return to deal with the system (1.1) and a key role in our approach will be played by an analysis of the coupled estimate $\int_{\Omega}(u+1)^p + \int_{\Omega}|\nabla v|^{2q}$, which is easier than the method of Maximal Sobolev Regularity in [15].