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Blowup of a free boundary problem with a nonlocal reaction term☆

Jian Yang

School of Mathematical Sciences, University of Jinan, Shandong 250022, China

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ABSTRACT

We consider a blowup problem of a reaction–diffusion equation with a nonlocal reaction term. Such a problem arises in the description of the species inhabiting in a region surrounded by an inhospitable area with the free boundary representing the spreading front of the species. Firstly, we give some sufficient conditions for finite time blowup. Then we show that the solution decays at an exponential rate and the two free boundaries converge to a finite limit provided the initial data is small and the result is different for the positive and non-positive growth rate.

1. Introduction

In this paper, we study the following reaction–diffusion equation with a nonlocal reaction term

$$\begin{cases} u_t = du_{xx} + u\left(a + b\displaystyle\int_{g(t)}^{h(t)} u^p dx - u^{q-1}\right), & g(t) < x < h(t),\ t > 0, \\ u(t, g(t)) = 0,\ g'(t) = -\mu u_x(t, g(t)), & t > 0, \\ u(t, h(t)) = 0,\ h'(t) = -\mu u_x(t, h(t)), & t > 0, \\ -g(0) = h(0) = h_0,\ u(0, x) = u_0(x), & -h_0 \le x \le h_0, \end{cases} \tag{1.1}$$

where $x = g(t), h(t)$ are free boundaries to be determined, $a \in \mathbb{R}$, $p, q \ge 1$ and h_0, μ, d, b are some given positive constants. The initial function u_0 is chosen from $\mathscr{X}(h_0)$ for some $h_0 \in (0, \infty)$, where

$$\mathscr{X}(h_0) := \left\{u_0 \in C^2([-h_0, h_0]) :\ u_0 > 0 \text{ in } (-h_0, h_0) \text{ with } u_0(-h_0) = u_0(h_0) = 0\right\}. \tag{1.2}$$

In the problem (1.1), $u(t, x)$ may represent the population density of a species over a one dimensional habitat and the initial function $u_0(x)$ stands for the population of the species which occupy an initial region

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E-mail address: yangjian86419@126.com.

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$(-h_0, h_0)$. This model may be used to describe some species inhabiting a region surrounded by an inhospitable area since Dirichlet boundary conditions are introduced. Here a is the growth rate of the species, the term $-u^{q-1}$ describes the limiting effect of crowding in the population, that is the competition of the individuals of the species for the resources of the environment. The size of q indicates the intensity of competition. The nonlocal reaction term has some specific meaning: $b > 0$ means the individuals cooperate globally to survive; when $b = 0$, the nonlocal term has no effect on this model and $(1.1)_1$ becomes the classical logistic equation with $q = 2$. Such problems with positive growth rate have been systematically studied by some authors, please see the details in references [1–3] and so on. Taking into account the significance of the study, we are only concerned with the case $b > 0$ in this article. In the process of the population range expansion, the free boundary conditions $(1.1)_2$, $(1.1)_3$ indicate that the individual of species enters the unpopulated environment but the environment is unfavorable for their survival. In real life, this is more reasonable to tend the free boundary condition in the model, the free boundary here are used to describe the species that are struggling to expand their territory to obtain more chance of survival and there are a lot of practical examples. For instance, if people live in an area near the sea, they may do some land reclamation. Also, some residents living in the oasis can make the desert green or develop waste land and so on.

As we know, the free boundary condition has been used in many areas, for example, it may model the wound healing [4], the melting of ice [5], and the spreading of species [1–3,6,7].

In the past few years, many scholars investigated the blowup problems and obtain some rich results. In [8–11], the authors studied the blowup phenomenon of reaction–diffusion equation with a free boundary, they proved the solution blows up in finite time in L^∞ norm and all global solutions are bounded and decay uniformly to 0. In [12], the authors considered the blowup problem in a bounded domain in higher dimension and they gave the set of blowup points. Moreover, in [13], the authors studied the blowup phenomenon of Cauchy problem in $\mathbb{R}^n$. With some suitable assumptions and simple conditions, they obtained the existence and nonexistence, large time behavior or life span of the solution.

In this paper, we are interested in the blowup phenomenon of the solution and the set of blowup points of (1.1). Since the solution of the problem (1.1) may blow up in finite time, we shall use the following notation:

$$T^* = T^*(u_0) := \sup\{t > 0 : \text{ the classical solution exists on } [0, t] \text{ for the initial data } u_0\}.$$

If $T^* < \infty$ and

$$\lim_{t\to T^*} \|u(t, \cdot)\|_{L^\infty([g(t),h(t)])} = \infty,$$

then we say that the solution u blows up in finite time and T^* is known as the blowup time. In Section 3 we will see that as long as $T^* < \infty$, then the solution must blow up in finite time.

The main purpose of this paper is to investigate the long time behavior of the solutions of (1.1) provided the initial datum has compact supports. This paper is organized as follows: In Section 2, we present some basic results which are fundamental for this research. In Section 3, by applying the comparison principle and the construction of suitable upper and lower solutions, we give a number of sufficient conditions for blowup and vanishing.

2. Some basic results

In this section we give some basic results which will be frequently used later in this paper. The following local existence result can be proved by the same method as in [1,4,14].

Theorem 2.1. *For any given $u_0 \in \mathscr{X}(h_0)$ and any $\alpha \in (0, 1)$, there is a $T > 0$ such that the problem (1.1) admits a unique solution*

$$(u, g, h) \in C^{(1+\alpha)/2,1+\alpha}(\overline{G}_T) \times C^{1+\alpha/2}([0, T]) \times C^{1+\alpha/2}([0, T]).$$