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Complex Ginzburg–Landau equations with dynamic boundary conditions

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ABSTRACT

The initial-dynamic boundary value problem (idbvp) for the complex Ginzburg–Landau equation (CGLE) on bounded domains of $\mathbb{R}^N$ is studied by converting the given mathematical model into a Wentzell initial–boundary value problem (ibvp). First, the corresponding linear homogeneous idbvp is considered. Secondly, the forced linear idbvp with both interior and boundary forcings is studied. Then, the nonlinear idbvp with Lipschitz nonlinearity in the interior and monotone nonlinearity on the boundary is analyzed. The local well-posedness of the idbvp for the CGLE with power type nonlinearities is obtained via a contraction mapping argument. Global well-posedness for strong solutions is shown. Global existence and uniqueness of weak solutions are proven. Smoothing effect of the corresponding evolution operator is proved. This helps to get better well-posedness results than the known results on idbvp for nonlinear Schrödinger equations (NLS). An interesting result of this paper is proving that solutions of NLS subject to dynamic boundary conditions can be obtained as inviscid limits of the solutions of the CGLE subject to same type of boundary conditions. Finally, long time behavior of solutions is characterized and exponential decay rates are obtained at the energy level by using control theoretic tools.

1. Introduction

This article is devoted to the analysis of the initial-dynamic boundary value problem (idbvp) for the complex Ginzburg–Landau equation (CGLE):

$$\begin{cases} u_t - (\lambda + i\alpha)\triangle u + f(u) = 0 & \text{in } \Omega \times \mathbb{R}_+, \\ \dfrac{\partial u}{\partial \nu} = -g(u_t) & \text{on } \Gamma_1 \times \mathbb{R}_+, \\ u = 0 & \text{on } \Gamma_0 \times \mathbb{R}_+, \\ u(0) = u_0 & \text{in } \Omega. \end{cases} \tag{1.1}$$

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In (1.1), $\Omega \subset \mathbb{R}^N$ is a bounded regular domain with boundary Γ, which is the union of Γ_0 and Γ_1, two nonempty, non-intersecting, connected $(n-1)$-dimensional manifolds. $u = u(x,t)$ is a complex valued function that denotes the complex oscillation amplitude. $f(u)$ will be defined either as the usual power type nonlinearity $f(u) = (\kappa + i\beta)|u|^{p-1}u - \gamma u$ (Sections 6–10) or as an appropriate Lipschitz function (Section 5). Here, $\beta \in \mathbb{R}$ and $\alpha \in \mathbb{R}$ are the (nonlinear) frequency and (linear) dispersion parameters, respectively. Without loss of generality, α can be taken as positive. Therefore we will also assume $\alpha > 0$ throughout the text. β can have both signs except when we discuss global solutions with power type nonlinearities, where it will be assumed to be positive. $\frac{\partial u}{\partial \nu}$ denotes the unit outward normal derivative. $p \geq 2$ is the source power index. Other parameters satisfy $\lambda, \kappa > 0, \gamma \in \mathbb{R}$. $g : \mathbb{C} \to \mathbb{C}$ is a complex valued function which is either taken as identity (Sections 2–4, 6–10) or as a monotone function (Section 5) satisfying suitable growth conditions to be specified later in Assumption 2.1.

CGLE is a fundamental model in mathematical physics to describe near-critical instability waves, such as a reaction diffusion system near a Hopf-bifurcation. Concrete applications of this equation include nonlinear waves, second-order phase transitions, superconductivity, superfluidity, Bose–Einstein condensation, and liquid crystals. See [1] and the references therein for an overview of several phenomena described by the CGLE.

CGLE simultaneously generalizes the nonlinear heat and nonlinear Schrödinger equations (NLS), both of which can be obtained in the limit as the parameter pairs (α, β) and (λ, κ) tend to zero, respectively. Therefore, it is natural to expect that CGLE carries some of the characteristics of the nonlinear heat equation and NLS. The latter two types of equations have been studied to some extent under dynamic boundary conditions. However, there has been no such progress for the CGLE. Most models assumed ideal set-ups neglecting possible linear and nonlinear interior–boundary interactions. See for example [2–12], and [13] for existence and non-existence results on the CGLE in the case of the whole space or domains with homogeneous or periodic boundary conditions. There are only a few results on the CGLE under nonhomogeneous boundary conditions [14–18]. The NLS subject to inhomogeneous or nonlinear boundary conditions, which can be considered a limiting case of the CGLE, took much more attention in recent years; see for example [19–28], [29–32], and [33].

Recently, [34] studied the defocusing cubic Schrödinger equation with dynamic boundary conditions on a bounded domain $\Omega \subset \mathbb{R}^N$ with smooth boundary for $N = 2, 3$. The model considered in [34] was the special case of the problem (1.1) where $\alpha = \beta = 1$ and $\lambda = \kappa = \gamma = 0$. In this work, the authors obtained the local well-posedness of strong (H^2) solutions for $N = 2, 3$ and global well-posedness of strong solutions for $N = 2$. In addition, the existence (without uniqueness) of weak (H^1) solutions was obtained for $N = 2, 3$. Moreover, it was proven that the energy of the weak solutions satisfies a uniform decay rate estimate under appropriate monotonicity conditions imposed on the nonlinear term appearing in the dynamic boundary conditions.

The key idea in [34] is replacing the given dynamic boundary condition with an equivalent boundary condition, which is obtained by replacing u_t on the boundary with the Laplacian and other terms coming from the main equation. This enables one to obtain the generation of a semigroup in an appropriate topology. The idea of using a boundary condition which involves the trace of the Laplacian comes from Venttsel's work [35]. In his paper, Venttsel was interested in finding the most general boundary condition which restricts the closure of a given elliptic operator to the infinitesimal generator of a semigroup of positive contraction operators on the Banach space of continuous functions over a regular compact region [35]. The result of this work was the discovery of the generalized Venttsel (more commonly "Wentzell") boundary condition $a\Delta u + b\frac{\partial u}{\partial \nu} + cu = 0$ on Γ, which provided the desired property for $a > 0, b, c \geq 0$.

Physically, this boundary condition can be considered as a (damped) harmonic oscillator acting at each point on the boundary. In the case of the heat equation, this means that the boundary can act as a heat source or sink depending on the physical situation. These boundary conditions also arise naturally in the