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Nonlinear Hartree equation in high energy-mass

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ABSTRACT

This paper is concerned with the Cauchy problem of the nonlinear Hartree equation. By constructing a constrained variational problem, we get a refined Gagliardo–Nirenberg inequality and the best constant for this inequality. We thus derive two conclusions. Firstly, by establishing and analyzing the invariant manifolds, we obtain a new criteria for global existence and blowup of the solutions. Secondly, we get other sufficient condition for global existence with the discussing of the Bootstrap argument. And based on these two conclusions, we also deduce so-called energy-mass control maps, which expose the relationship between the initial data and the solutions.

1. Introduction

Bose–Einstein condensation (BEC) in gases with weak attractive two-body interactions can be found in system of ^{7}Li atoms as long as the gas in the trap has a sufficiently low density [1,2]. By the heuristic discussion, we have that the dynamical evolution of the bosonic system in its mean-field regime is described by nonlinear Hartree equations. Thus, the aim of this paper is to study the following Hartree equation:

$$\begin{cases} i\varphi_t + \Delta\varphi + (V(x) * |\varphi|^2)\varphi = 0, \\ \varphi(0,x) = \varphi_0, \quad x \in \mathbb{R}^N \end{cases} \tag{1.1}$$

where $\varphi(t,x) : \mathbb{R}^+ \times \mathbb{R}^N \longrightarrow \mathbb{C}$ is a complex function, Δ is Laplacian operator on $\mathbb{R}^N, V(x) = \frac{1}{|x|^\alpha}$ $(0 < \alpha < \min\{N,4\})$ and $*$ is the standard convolution in $\mathbb{R}^N$. The initial data $\varphi_0 \in H^1(\mathbb{R}^N)$, and $H^1(\mathbb{R}^N) = W^{1,2}(\mathbb{R}^N)$ is the standard Sobolev space with the norm $\|\varphi\|_{H^1}^2 = \int_{\mathbb{R}^N}(|\nabla\varphi|^2 + |\varphi|^2)dx$.

The success of experiment in atomic BEC has stimulated great interest in the properties of Cauchy problem (1.1). For instance, Cazenave [3] stated the local well-posedness of Cauchy problem (1.1) in $H^1(\mathbb{R}^N)$

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when $0 < \alpha < \min\{N, 4\}$ Lions [4] studied the existence of the standing wave of Cauchy problem (1.1) for $\alpha = 1$ (mass subcritical case). Miao [5] showed the existence of the standing waves and the blow up criterion of Cauchy problem (1.1) for L^2 critical (i.e., $\alpha = 2$). And for L^2 supercritical (i.e., $2 < \alpha < \min\{N, 4\}$), Wang [6] and Chen and Guo [7] discussed strong instability of the standing waves of Cauchy problem (1.1) in supercritical case with and without harmonic potential respectively.

In this paper, we plan to discuss the Cauchy problem (1.1) for L^2 supercritical. As we mentioned, Wang [6] and Chen and Guo [7] had discussed this issue before, however. We notice that in [6,7], they obtained the condition for blowup of Cauchy problem (1.1) by constructing a type of cross constrained variational problem and some of these functionals haven't clear physical meanings. Can we use the energy and mass to character the criterion for global existence and blowup? Cazenave also mentioned this topic in their monographs [3]. At the meantime, a collapse processing is observed in the experiments if the number of ^{7}Li atoms in the trap exceeds some threshold value. So this problem is also pursued strongly in Physics (see [8] and the references therein). This is one of the motivations for discussing this problem.

However, in spite of quite a number of contributions dealing with this problem (see [9–12] as well as the other relevant references), almost all of them still need to restrict the energy-mass functional(or some other functionals)to be less than its minimum in one subset. What will happen when the energy-mass is less than the minmax value of the energy-mass functional in the whole space? Obviously, this minmax value is larger than its minimum in one subset. Consequently, we also can answer the question: what will happen when the energy-mass functional is less than a number which is larger than the minimum in one subset? To fill this gap, we do more detailed discussion for the properties of the equation and energy-mass functional and extend the argument to the case that we mentioned before. Actually, we get a sharp condition for the global existence of the solution in higher energy-mass. In addition, these conditions are precisely computed.

Finally, we expect to find some details of the energy-mass criteria, which is called energy-mass control map. This control map shows the relationships between the initial data and some behaviors of the solutions corresponding to these initial data. Furthermore, we find a set which indicate both global existence and blowup. To get more information of this set, we divide it into two parts, and then strip a global existence subset from one of the parts. In fact, to our knowledge no results have been already known in this direction.

The plan of this paper is as follows. In Section 2, we give some concerned preliminaries and obtain a refined Gagliardo–Nirenberg inequality. And in Section 3, we give a new criterion of global existence and blowup for Cauchy problem (1.1). And the other sufficient condition for global existence is also obtained by Bootstrap argument. In the last section, we derive the so-called energy-mass control map by discussing the relations of the results which is obtained in the Section 3.

2. Preliminaries

2.1. Notations and some known lemmas

In this subsection, we will give some concerned preliminaries. Throughout this paper, C denotes various positive constants. For simplicity, here and hereafter, we denote $\|\cdot\|_p$ to denote the norm of $L^p(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} \cdot dx$ by $\int \cdot dx$ unless stated otherwise. Moreover, we denote $\Sigma := \{u \in H^1(\mathbb{R}^N) : |x|u \in L^2(\mathbb{R}^N)\}$ and

$$\|\varphi\|_V = \left(\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\varphi(x)|^2|\varphi(y)|^2}{|x-y|^\alpha} dxdy\right)^{\frac{1}{4}}. \tag{2.1}$$

Now, we begin by recalling the Hardy–Littlewood–Sobolev inequality.