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Global in time existence of self-interacting scalar field in de Sitter spacetimes

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ABSTRACT

We prove the existence of a global in time solution of the semilinear Klein–Gordon equation in the de Sitter space–time. The coefficients of the equation depend on spatial variables as well, that make results applicable to the space–time with the time slices being Riemannian manifolds.

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0. Introduction

In this paper we prove the existence of a global in time solution of the semilinear Klein–Gordon equation in the de Sitter space–time. The coefficients of the equation depend on spatial variables as well, that make results applicable to the space–time with the time slices being Riemannian manifolds. In the spatially flat de Sitter model, these slices are $\mathbb{R}^3$, while in the spatially closed and spatially open cases these slices can be the three-sphere $\mathbb{S}^3$ and the three-hyperboloid $\mathbb{H}^3$, respectively (see, e.g., [1, p. 113]).

The metric g in the de Sitter space–time is defined as follows, $g_{00} = g^{00} = -1, g_{0j} = g^{0j} = 0, g_{ij}(x,t) = e^{2t}\sigma_{ij}(x), i,j = 1,2,\dots,n$, where $\sum_{j=1}^{n} \sigma^{ij}(x)\sigma_{jk}(x) = \delta_{ik}$, and δ_{ij} is Kronecker's delta. The metric $\sigma^{ij}(x)$ describes the time slices. In the quantum field theory the matter fields are described by a function ψ that must satisfy equations of motion. In the case of a massive scalar field, the equation of motion is the semilinear Klein–Gordon equation generated by the metric g:

$$\square_g \psi = m^2\psi + V'_\psi(x,\psi).$$

Here m is a physical mass of the particle. In physical terms this equation describes a local self-interaction for a scalar particle. A typical example of a potential function would be $V(\psi) = \psi^4$. The semilinear equations are also commonly used models for general nonlinear problems.

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The covariant Klein–Gordon equation in the de Sitter space–time in the coordinates is

$$\psi_{tt} - \frac{e^{-2t}}{\sqrt{|\det \sigma(x)|}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \left(\sqrt{|\det \sigma(x)|} \sigma^{ij}(x) \frac{\partial}{\partial x^j} \psi \right) + n\psi_t + m^2 \psi = F(\psi).$$

This is a special case of the equation

$$\psi_{tt} + n\psi_t - e^{-2t} A(x, \partial_x)\psi + m^2 \psi = F(\psi),$$

where $A(x, \partial_x) = \sum_{|\alpha| \le 2} a_\alpha(x) \partial_x^\alpha$ is a second order partial differential operator. More precisely, in this paper we assume that $a_\alpha(x), |\alpha| = 2$, is positive definite (and symmetric).

In [2–4] a global existence of small data solutions of the Cauchy problem for the semilinear Klein–Gordon equation and systems of equations in the de Sitter space–time with flat time slices, that is, $\sigma^{ij}(x) = \delta^{ij}$, is proved. The nonlinearity F was assumed Lipschitz continuous with exponent $\alpha \ge 0$ (see definition below). It was discovered that unlike the same problem in the Minkowski space–time, no restriction on the order of nonlinearity is required, provided that a physical mass of the field belongs to some set, $m \in (0, \sqrt{n^2 - 1}/2] \cup [n/2, \infty)$. For $n = 3$ the mass m interval $(0, \sqrt{2})$ is called the Higuchi bound in quantum field theory [5]. The proof of the global existence in [2–4] is based on the special integral representations (see Section 1) and $L^p - L^q$ estimates.

In the present paper we are going to extend the small data global existence result of [4] for the spatially flat de Sitter space–time to the de Sitter space–time with the time slices being Riemannian manifolds.

To formulate the main theorem of this paper we need the following description of the nonlinear term. Let $B_p^{s,q}$ be the Besov space.

Condition ($\mathcal{L}$). *The function F is said to be Lipschitz continuous with exponent $\alpha \ge 0$ in the space $B_p^{s,q}$ if there is a constant $C \ge 0$ such that*

$$\|F(x, \psi_1(x)) - F(x, \psi_2(x))\|_{B_p^{s,q}} \le C \|\psi_1 - \psi_2\|_{B_{p'}^{s,q}} \left(\|\psi_1\|_{B_{p'}^{s,q}}^\alpha + \|\psi_2\|_{B_{p'}^{s,q}}^\alpha \right) \tag{0.1}$$

for all $\psi_1, \psi_2 \in B_{p'}^{s,q}$, where $1/p + 1/p' = 1$.

For the case of $B_2^{s,2} = H_{(s)}(\mathbb{R}^n)$, define the complete metric space

$$X(R, s, \gamma) := \{\psi \in C([0, \infty); H_{(s)}(\mathbb{R}^n)) \mid \|\psi\|_X := \sup_{t \in [0,\infty)} e^{\gamma t} \|\psi(x, t)\|_{H_{(s)}(\mathbb{R}^n)} \le R\},$$

$\gamma \ge 0$, with the metric

$$d(\psi_1, \psi_2) := \sup_{t \in [0,\infty)} e^{\gamma t} \|\psi_1(x, t) - \psi_2(x, t)\|_{H_{(s)}(\mathbb{R}^n)}.$$

We denote $\mathcal{B}^\infty$ the space of all $C^\infty(\mathbb{R}^n)$ functions with uniformly bounded derivatives of all orders.

Theorem 0.1. *Let $A(x, \partial_x) = \sum_{|\alpha| \le 2} a_\alpha(x) \partial_x^\alpha$ be a second order negative elliptic differential operator with real coefficients $a_\alpha \in \mathcal{B}^\infty$. Assume that the nonlinear term F is Lipschitz continuous with exponent $\alpha > 0$ in the space $H_{(s)}(\mathbb{R}^n), s > n/2 \ge 1$, and $F(x, 0) \equiv 0$. Assume also that $m \in (0, \sqrt{n^2 - 1}/2] \cup [n/2, \infty)$. Then there exists $\varepsilon_0 > 0$ such that, for every given functions $\psi_0, \psi_1 \in H_{(s)}(\mathbb{R}^n)$, such that*

$$\|\psi_0\|_{H_{(s)}(\mathbb{R}^n)} + \|\psi_1\|_{H_{(s)}(\mathbb{R}^n)} \le \varepsilon, \quad \varepsilon < \varepsilon_0,$$

there exists a global solution $\psi \in C^1([0, \infty); H_{(s)}(\mathbb{R}^n))$ of the Cauchy problem

$$\psi_{tt} + n\psi_t - e^{-2t} A(x, \partial_x)\psi + m^2 \psi = F(x, \psi), \tag{0.2}$$

$$\psi(x, 0) = \psi_0(x), \qquad \psi_t(x, 0) = \psi_1(x). \tag{0.3}$$