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On an integrable Camassa–Holm type equation with cubic nonlinearity

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A B S T R A C T

We discuss an integrable Camassa–Holm type equation with cubic nonlinearity. Asymptotic profile has been shown in the sense that strong solutions arising from algebraic decaying initial data will keep this property at infinity in the spatial variable in its lifespan. Moreover, for a global solution, measure of potential support is presented.

1. Introduction

We consider the following Camassa–Holm type equation

$$u_t - u_{xxt} + 4u^2 u_x = 3uu_x u_{xx} + u^2 u_{xxx} \tag{1.1}$$

which was recently discovered in a symmetry classification on nonlocal PDEs with quadratic or cubic nonlinearity by Vladimir Novikov [1]. The perturbative symmetry approach yields necessary conditions for a PDE to admit infinitely many symmetries. Using this approach, Novikov was able to isolate (1.1) to find its first few symmetries, and subsequently found a scalar Lax pair for it, then proved that (1.1) is integrable. Taking convolution with Green's function $G(x) = e^{-|x|}/2, x \in \mathbb{R}$ for the operator $(1-\partial_x^2)^{-1}$ gives the following equivalent nonlocal form

$$u_t + u^2 u_x + G * (3uu_x u_{xx} + 2u_x^3 + 3u^2 u_x) = 0. \tag{1.2}$$

By comparison with the celebrated Camassa–Holm equation [2]

$$u_t + uu_x + \partial_x G * \left(u^2 + \frac{1}{2}u_x^2\right) = 0,$$

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it is easy to observe that (1.1) (or (1.2)) has nonlinear terms that are cubic rather than quadratic. It is then viewed as a generalization of Camassa–Holm equation. Hone and Wang [3] have shown that Eq. (1.1) admits peakon solutions like the Camassa–Holm equation, moreover, they gave a matrix Lax pair for (1.1), and showed how it was related by a reciprocal transformation to a negative flow in the Swasa-Kotera hierarchy. Infinitely many conserved quantities were found as well as a bi-Hamiltonian structure. Then Hone, Lundmark and Szmigielski [4] calculated the explicit formulas for multipeakon solutions of (1.1) using the matrix Lax pair found by Hone and Wang.

Like the Camassa–Holm equation, among those infinitely many conservation laws, the most important quantity conserved by a solution u to (1.1) is its H^1-norm

$$\|u(x,t)\|_{H^1}^2 = \int_{\mathbb{R}} (u^2 + u_x^2) dx,$$

which also gives a bound of u

$$\|u(x,t)\|_{L^\infty}^2 \le \frac{1}{2}\|u(x,t)\|_{H^1}^2 = \frac{1}{2}\|u_0\|_{H^1}^2 \triangleq \mathcal{C}. \tag{1.3}$$

Since the pioneering work made by Novikov, Hone, Wang, Lundmark et al. (1.1) has attracted much attention. Ni and Zhou [5] proved that the Cauchy problem for (1.1) is locally well-posed in Besov spaces $B^s_{2,r}$ with the critical index $s = 3/2$ and in Sobolev spaces H^s with $s > 3/2$ with the aid of Kato's semigroup theory. Blow-up phenomenon is also addressed by establishing various criteria [6,7]. Very recently, Tiglay [8] has obtained a global in time result when $(1 - \partial_x^2)u_0$ does not change sign, global weak solution was also shown by Wu [9] and [10,11]. The discussions on the initial value problem on the line and on the circle were referred to [12,8].

In this paper, two issues are addressed. One is to investigate the asymptotic behavior of strong solutions in the spatial variable with suitable decaying initial data. The other one is to give a reasonable estimate of potential support provided that strong solution exists globally.

2. Asymptotic profile

This section devotes to the investigation of asymptotic profile of solutions with algebraic decaying initial data. Precisely, we give an asymptotic description how the solutions behave under algebraic decaying initial value in spatial variable at infinity. This motivation derives from the recent work [5] where the authors have shown that the corresponding solution of (1.1) and its first-order spatial derivative retain exponential decay at infinity as what their initial values do. After all, the exponential decay of initial value is a faster way, then we try to consider the algebraic decaying initial data. We show the strong solution of (1.1) arising from initial data with a slower algebraically decaying way will keep this behavior in x-variable at infinity in its lifespan. This is the following theorem.

Theorem 2.1. *Assume that for some $T > 0$ and $s > 3/2, u(x,t) \in C([0,T]; H^s(\mathbb{R}))$ is a strong solution of (1.1) with its initial value $u_0(x) = u(x,0)$ satisfying*

$$|u_0(x)| \sim O(x^{-\theta}) \text{ and } |u_{0x}(x)| \sim o(x^{-\theta}) \quad \text{as } x \uparrow \infty$$

for some $\theta \in (1/2, 1]$. Then

$$|u(x,t)| \sim O(x^{-\theta}) \text{ and } |u_x(x,t)| \sim o(x^{-\theta}) \quad \text{as } x \uparrow \infty,$$

uniformly in the time interval $[0,T]$.