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Multiple solutions of superlinear cooperative elliptic systems at resonant☆

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ABSTRACT

In this paper, we study a class of resonant cooperative elliptic systems with nonlinearities being superlinear at infinity. We remove some classical conditions near 0 used before by many authors, and we obtain infinitely many nontrivial solutions for this class of systems, a topic about which very little is known. Our main result extends and improves some recent results in the literature.

1. Introduction and main results

In this paper, we study the following nonlinear elliptic system

$$\begin{cases} -\triangle u = \lambda u + \delta v + f(x,u,v) & \text{in } \Omega, \\ -\triangle v = \delta u + \gamma v + g(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbf{R}^N (N \geq 3)$ and $\lambda, \delta, \gamma \in \mathbf{R}$. The nonlinearities (f, g) of (1.1) are the gradient of some function $F(x,U)$, i.e., $\nabla F(x,U) = (f,g)$, where $\nabla F(x,U)$ denotes its gradient with respect to the $U = (u,v)$ variable.

The system (1.1) is related to the reaction–diffusion systems appearing in the chemical and biological phenomena, and it is called a *cooperative elliptic system* by [1,2]. The system (1.1) is called *resonant* if the following condition holds:

(V_1) $\sigma(A^*) \cap \sigma(-\triangle) \neq \emptyset$,

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where

$$A^* = \begin{pmatrix} \lambda & \delta \\ \delta & \gamma \end{pmatrix}; \qquad \sigma(A^*) = \{\xi, \zeta\} = \left\{ \frac{\lambda+\gamma}{2} \pm \sqrt{\left(\frac{\lambda-\gamma}{2}\right)^2 + \delta^2} \right\}$$

denotes the spectrum of the matrix A^* and $\sigma(-\triangle) = \{\lambda_k : \ k = 1, 2, \cdots \ \text{and} \ 0 < \lambda_1 < \lambda_2 < \cdots\}$ denotes the eigenvalues of the Laplacian on Ω with zero boundary condition.

In the past decades, the system (1.1) has been studied by some authors [1–11]. By minimax techniques, the authors [1] established the variational structure of (1.1) and proved that (1.1) has *one* solution in the case where the nonlinearity $F(x, U)$ is *subquadratic* at infinity, i.e.,

$$\lim_{|U| \to \infty} \frac{F(x, U)}{|U|^2} = 0 \quad \text{uniformly for } x \in \Omega.$$

If $F \in C^1(\bar{\Omega} \times \mathbf{R}^2, \mathbf{R})$, Ma [7] obtained infinitely many solutions for (1.1) with $F(x, U)$ being *subquadratic* at infinity by the minimax techniques. If $F \in C^1$, the authors [8,10] obtained *one solution or two solutions* for (1.1) with $F(x, U)$ being *subquadratic* at infinity by the computations of the critical groups and the Morse theory. The results in [8] are based on the following conditions:

$$\lim_{|U| \to 0} \frac{|\nabla F(x, U)|}{|U|^\theta} = 0 \quad \text{or} \quad \limsup_{|U| \to 0} \frac{|\nabla F(x, U)|}{|U|^\theta} < +\infty \quad \text{uniformly for } x \in \Omega, \quad 1 < \theta < 2^*/2. \tag{1.2}$$

The results in [10] are based on the following condition:

$$\lim_{|U| \to 0} \frac{|\nabla F(x, U)|}{|U|} = 0 \quad \text{uniformly for } x \in \Omega. \tag{1.3}$$

If $F \in C^2(\bar{\Omega} \times \mathbf{R}^2, \mathbf{R})$ and

$$F(x, 0) = 0, \quad \forall x \in \Omega, \tag{1.4}$$

Pomponio [2] considered the case where $\lim_{|u|+|v| \to \infty} F''(x, u, v) = 0$ ($F \in C^2(\bar{\Omega} \times \mathbf{R}^2, \mathbf{R})$ and F'' is the Hessian matrix with respect to (u, v)) and proved that (1.1) has *one* or $N-1$ nontrivial solutions by using a penalization technique and the Morse theory. If (1.4) holds, $F \in C^1$ and some other conditions hold, Zou [9] obtained infinitely many solutions of (1.1) with $F(x, U)$ being *subquadratic* at infinity by the methods used in [12]. Zou [11] also obtained infinitely many solutions of (1.1) with $F(x, U)$ being *subquadratic* at infinity and with some conditions near $U = 0$ (In fact, the conditions near $U = 0$ also imply that (1.4) holds). In the whole space $\mathbf{R}^N$, Chen and Ma [3] studied an elliptic system different from (1.1), and they [3] obtained the *existence* of nontrivial solutions for the system with nonlinearities being *asymptotically linear* and *superlinear* at infinity by variational methods.

Note that the above results of (1.1) are all about the case where $F(x, U)$ is *subquadratic* at infinity. However, the case where the nonlinearity $F(x, U)$ is *superquadratic* at infinity, about which very little is known. Recently, some authors [4–6,13] considered the case where $F(x, U)$ is *superquadratic* at infinity. For a special case of (1.1) with $\delta = 0$, Chen [4,5] obtained the existence [4] and multiplicity [5] of solutions for (1.1) by a variant weak linking theorem [14] and a generalized Nehari manifold method [15], respectively. We should mention that the results in [4,5] are all based on the condition (1.3) and the following condition:

$$(\nabla F(x, U), U) > 2F(x, U) > 0, \quad \forall U \in \mathbf{R}^2 \backslash \{0\}, \ \forall x \in \Omega. \tag{1.5}$$

Later, Chen and Ma [6,13] removed the conditions (1.3) and (1.5), and they obtained infinitely many nontrivial solutions of the general system ($\delta \not\equiv 0$) (1.1) with $F(x, U)$ being *subquadratic* or *superquadratic* at infinity by two variant fountain theorems [16]. Recently, the authors [17] obtained the existence of infinitely many solutions of (1.1) with $F(x, U)$ being *subquadratic* as $|U| \to 0$. Similar results are also obtained by the authors [18] in the study of periodic solutions for second order Hamiltonian systems.