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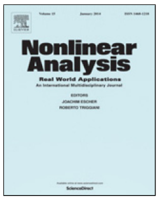

Hamiltonian model for coupled surface and internal waves in the presence of currents

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A B S T R A C T

We examine a two dimensional fluid system consisting of a lower medium bounded underneath by a flatbed and an upper medium with a free surface. The two media are separated by a free common interface. The gravity driven surface and internal water waves (at the common interface between the media) in the presence of a depth-dependent current are studied under certain physical assumptions. Both media are considered incompressible and with prescribed vorticities. Using the Hamiltonian approach the Hamiltonian of the system is constructed in terms of 'wave' variables and the equations of motion are calculated. The resultant equations of motion are then analysed to show that wave–current interaction is influenced only by the current profile in the 'strips' adjacent to the surface and the interface. Small amplitude and long-wave approximations are also presented.

1. Introduction

It has been known for many centuries that the ocean contains currents that flow along generally consistent paths. The Spanish galleons transporting gold and silver from Mexico to Spain made use of the Gulf Stream to help them return home. Since then, scientists have gained much more information on both where currents flow and why. In the oceans currents very often exist with undercurrents. The first undercurrent was discovered in 1951 by Townsend Cromwell who was investigating fishing techniques in the central Pacific Ocean. Undercurrents have since been found under most major currents. The equatorial region in the Pacific is characterised by a thin shallow layer of warm and less dense water over a much deeper layer of cold denser water. The two layers are separated by a sharp *thermocline* (where the temperature gradient has a maximum, it is very close to the *pycnocline*, where the pressure gradient has a maximum) at a depth, depending on

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the location, but usually at 100–200 m beneath the surface. For modelling purposes both layers are assumed homogeneous with a sharp boundary at the thermocline/pycnocline (see [1]).

The Equatorial Undercurrent (EUC) flows in a region that is roughly within 200–300 km (below 3° latitude) of the Equator, it is symmetric about the Equator and extends nearly across the whole length (more than 12000 km) of the Pacific Ocean basin [2]. With speeds in excess of 1 m/s, the EUC is one of the fastest permanent currents in the world.

The flow has nearly two-dimensional character, with small meridional variations. While at depths in excess of about 240 m there is, essentially, an abyssal layer of still water, the ocean dynamics near the surface is quite complex. In this region the wave motion typically comprises surface gravity waves with amplitudes of 1–2 m and oscillations with an amplitude of 10–20 m at the thermocline (of mean depth between 50 m and 150 m). These waves interact with the underlying currents. In that case the velocity is (anti-) parallel to the Earth's angular speed $\boldsymbol{\omega}$, so their vector product is zero. This feature distinguishes the dynamics of the equatorial zone from the ocean dynamics at higher latitudes.

The strong stratification confines the wind-driven currents to a shallow near-surface region, less than 200 m deep. In the Atlantic and Pacific, the westward trade winds induce a westward surface flow at speeds of 25–75 cm/s, while a jet-like current — the Equatorial Undercurrent (EUC) — flows below it towards the East (counter to the surface current), attaining speeds of more than 1 m/s at a depth of nearly 100 m. The wind-generated equatorial current in the layer above the thermocline is with a strictly monotonic depth-dependence and exhibits flow-reversal, while beneath the thermocline the current simply decays with increasing depth, being irrelevant in the abyssal region.

While viscous theory is essential in explaining the generation of the equatorial current induced by wind forcing, inviscid theory is adequate for the study of non-turbulent wave–current interactions since the relevant Reynolds numbers are very large (see [3]).

For some general facts concerning the description of waves interacting with currents we refer to the following reviews and monographs [4–7] and the references therein. The present study draws from previous single medium irrotational [8–12] and rotational [13,14,4,15–19] studies as well as from studies of two-media systems such as [20–38].

The Hamiltonian approach to water waves dynamics has been put forward for the first time by Zakharov [8]. The Hamiltonian formulation describing the two-dimensional nonlinear interaction between coupled surface waves, internal waves, and an underlying current with piecewise constant vorticity, in a two-media fluid overlying a flat bed has been developed in [26,27]. In the present study we will be following a similar approach, taking into account the shear current structure suggested in [24]. Related results for a flat surface (effectively rigid lid) has been studied in [28–31,39].

The model equations will be presented in a canonical Hamiltonian form and then small amplitude and long wave approximations will be derived.

2. Preliminaries

The system under study involves two-dimensional surface and internal gravity water waves and a depth dependent current as per Fig. 1.

The medium underneath the internal wave is defined by the domain

$$\Omega(\eta) = \{(x, y) \in \mathbb{R}^2 : -h < y < \eta(x, t)\}.$$

This medium is bounded at the bottom by an impermeable flatbed at a depth $-h$. The medium above the internal wave $y = \eta(x, t)$ is the domain

$$\Omega_1(\eta, \eta_1) = \{(x, y) \in \mathbb{R}^2 : \eta(x, t) < y < h_1 + \eta_1(x, t)\}.$$