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Optimal convergence rates for the strong solutions to the compressible Navier–Stokes equations with potential force

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ABSTRACT

In this paper, we consider the effect of external force on the large-time behavior of solutions to the Cauchy problem for the three-dimensional full compressible Navier–Stokes equations. We construct the global unique solution near the stationary profile to the system for the small $H^2(\mathbb{R}^3)$ initial data. Moreover, the optimal L^p–L^2 $(1 \le p \le 2)$ time decay rates of the solution to the system are established via a low frequency and high frequency decomposition.

1. Introduction

This paper is concerned with the Cauchy problem of the full compressible Navier–Stokes equations affected by the external potential force in $\mathbb{R}^3$:

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ \rho[u_t + (u \cdot \nabla)u] + \nabla P(\rho, \theta) = \mu \Delta u + (\mu + \mu')\nabla(\nabla \cdot u) + \rho F, \\ \rho c_V[\theta_t + (u \cdot \nabla)\theta] + \theta P_\theta(\rho, \theta)\nabla \cdot u = \kappa \Delta \theta + \Psi[u], \end{cases} \tag{1.1}$$

and the initial data

$$(\rho, u, \theta)(0, x) = (\rho_0, u_0, \theta_0)(x) \to (\rho_\infty, 0, \theta_\infty), \quad \text{as } |x| \to \infty. \tag{1.2}$$

Here the unknown functions $\rho > 0$, $u = (u_1, u_2, u_3)$, and θ denote the density, the velocity and the temperature; $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ is the space variable, $t > 0$ is the time variable; $P = P(\rho, \theta)$, μ, μ', $\kappa > 0$, and c_V are the pressure, the first and second viscosity coefficients, the coefficient of heat conduction,

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and the specific heat at constant volume, respectively. In addition, $F = F(x)$ is an external force and $\Psi = \Psi[u]$ is the dissipation function:

$$\Psi[u] = \frac{\mu}{2}\sum_{i,j=1}^{3}(\partial_i u_j + \partial_j u_i)^2 + \mu' \sum_{j=1}^{3}(\partial_j u_j)^2. \tag{1.3}$$

Throughout this paper, we assume that the above physical parameters satisfy $\mu > 0$ and $2\mu + 3\mu' \geq 0$ which deduce $\mu + \mu' > 0$. ρ_∞ and θ_∞ are positive constants, and $P(\rho,\theta)$ is smooth in a neighborhood of $(\rho_\infty, \theta_\infty)$ with $P_\rho(\rho_\infty,\theta_\infty) > 0$ and $P_\theta(\rho_\infty,\theta_\infty) > 0$.

In this work, we only consider the potential force, that is, $F = -\nabla\Phi(x)$. Under aforementioned assumptions, the existence of the stationary solution to the problem (1.1) and (1.2) has been established in [1]. The solution (ρ_*, u_*, θ_*) in a neighborhood of $(\rho_\infty, 0, \theta_\infty)$ is given by

$$\int_{\rho_\infty}^{\rho_*(x)} \frac{P_\rho(\eta,\theta_\infty)}{\eta}\mathrm{d}\eta + \Phi(x) = 0, \qquad u_*(x) = 0, \qquad \theta_*(x) = \theta_\infty, \tag{1.4}$$

and satisfies

$$\|\rho_* - \rho_\infty\|_{H^k(\mathbb{R}^3)} \leq C\|\Phi\|_{H^k(\mathbb{R}^3)}, \quad 0 \leq k \leq 4, \tag{1.5}$$

$$\sum_{k=1}^{4}\|(1+|x|)\nabla^k(\rho_* - \rho_\infty)\|_{L^2(\mathbb{R}^3)} \leq C\sum_{k=1}^{4}\|(1+|x|)\nabla^k\Phi\|_{L^2(\mathbb{R}^3)}. \tag{1.6}$$

We will construct the global unique solution to (1.1) near the steady state $(\rho_*, 0, \theta_\infty)$ when the initial perturbation belongs to the Sobolev space $H^2(\mathbb{R}^3)$. Our main results are stated as the following theorem.

Theorem 1.1. *Let $(\rho_0 - \rho_\infty, u_0, \theta_0 - \theta_\infty) \in H^2(\mathbb{R}^3)$, there exists some small constant $\varepsilon > 0$ such that if*

$$\|(\rho_0 - \rho_\infty, u_0, \theta_0 - \theta_\infty)\|_{H^2(\mathbb{R}^3)} + \|\Phi\|_{H^4(\mathbb{R}^3)} + \sum_{k=1}^{4}\|(1+|x|)\nabla^k\Phi\|_{L^2(\mathbb{R}^3)} \leq \varepsilon, \tag{1.7}$$

then the initial value problem (1.1) and (1.2) admits a unique solution (ρ, u, θ) globally in time which satisfies

$$\rho - \rho_* \in C^0([0,\infty); H^2(\mathbb{R}^3)) \cap C^1([0,\infty); H^1(\mathbb{R}^3)),$$
$$u, \theta - \theta_\infty \in C^0([0,\infty); H^2(\mathbb{R}^3)) \cap C^1([0,\infty); L^2(\mathbb{R}^3)).$$

Moreover, if the initial data $(\rho_0 - \rho_\infty, u_0, \theta_0 - \theta_\infty)$ is bounded in $L^p(\mathbb{R}^3)$ for any given $1 \leq p \leq 2$, the solution (ρ, u, θ) enjoys the following decay-in-time estimates:

$$\|\nabla(\rho - \rho_*, u, \theta - \theta_\infty)\|_{H^1(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{1}{2}} \quad \textit{for all } t \geq 0, \tag{1.8}$$

$$\|(\rho - \rho_*, u, \theta - \theta_\infty)\|_{L^q(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \quad \textit{for all } t \geq 0,\ 2 \leq q \leq 6, \tag{1.9}$$

$$\|\partial_t(\rho - \rho_*, u, \theta - \theta_\infty)\|_{L^2(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{1}{2}} \quad \textit{for all } t \geq 0, \tag{1.10}$$

for some positive constant C.

Remark 1.1. In Theorem 1.1, using the Sobolev imbedding inequalities in Lemma 2.1, (1.7) together with (1.5) and (1.6) yields

$$\|\rho_* - \rho_\infty\|_{H^4(\mathbb{R}^3)} + \sum_{k=1}^{3}\|(1+|x|)\nabla^k(\rho_* - \rho_\infty)\|_{L^2(\mathbb{R}^3)\cap L^3(\mathbb{R}^3)} \leq C\varepsilon. \tag{1.11}$$